Understanding the Number System

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UNDERSTANDING THE NUMBER SYSTEM

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The mathematical symbol used on the cover of this book is the familiar addition sign as it was written by the Renaissance mathematician and calculator Tartaglia. It is the first letter of the Italian *piu* (plus).

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Preface

Understanding the Number System is a substantial revision of earlier editions published as *Toward Improved Understanding of Mathematics, Part I*, and *Extending Mathematics Understanding*. The materials contained in these chapters were originally prepared as part of an extensive in-service education project designed to provide initial study of new mathematics for elementary school teachers. Since the first publication of these books in 1960, a large proportion of elementary teachers have extended their understanding of mathematics in a variety of ways. In-service education programs, college courses, lectures, and the use of new mathematics in the classroom have all contributed to the increased mathematical knowledge of elementary teachers. Elementary teachers are the first to admit, however, that the job of improving mathematics instruction in the schools is a continuing one which can be expected to occupy the attention of elementary teachers in the years ahead.

A large proportion of prospective elementary teachers completing their undergraduate work have credit in mathematics. As a group, they have experienced a wide range in the number of course credits and of course types. In some teacher education programs as much as twelve credits of mathematics are required; in others as few as two or three credits in methods meet the requirements. The mathematics courses vary widely also in their content and in the precision and vigor of the mathematics presented. This diversity in teacher preparation results in a wide variety of understanding on the part of teachers in our schools.

The present volume is designed to provide a treatment of mathematical topics in a manner which is mathematically precise but which can be easily read and understood by elementary teachers with a wide variety of backgrounds. Throughout the volume many of the mathematical discussions are set in the context of the elementary school and thereby directly relate to the experience and instructional needs of the teacher working with children in today's classrooms.

Understanding the Number System gives attention to the topics of sets, number and numeration systems, the properties and operations with both whole numbers and fractions, and with equivalence relations. *Extending Understandings of Mathematics*, a complementary volume, includes historical materials, measurement and geometry, statistics, and applications. As such it is designed to extend the understandings teachers have of the mathematics which they teach in the elementary school. Since the 1961 (combined) edition, major editorial revisions have been made in each chapter; new chapters in Geometry and in Classes of Numbers have been added; and the material is published as two paperbacks rather than as one casebound volume. The latter decision resulted from the belief that teachers' understandings have improved to a point where context texts will increasingly be needed as references used in close conjunction with methods texts. The methods courses which for the past several years too frequently have been devoted largely to content are now likely to assume basic content information and proceed from there with a thorough treatment of the problems of curriculum and instruction. Even though this assumption is made, it is understood that many students and many teachers will not have the content background needed to make the methods work meaningful, and these teachers need materials which will assist their individual efforts to build the necessary background. These two volumes are designed to assist these efforts.

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Contents

UNDERSTANDING THE NUMBER SYSTEM

Chapter 1

Concepts to be developed in this chapter are:

1. *There is a difference between numbers and numerals.*
2. *Numerals are symbols.*
3. *Number is a quantitative concept derived from perceptual images.*
4. *Numerals are used to represent this quantitative concept.*
5. *Number sense was developed in man before he invented numerals.*

1

Making Distinctions between Number and Numeral

CONTRASTING NUMERAL WITH NUMBER

Our everyday use of arithmetic results in a casual attitude toward the written symbols which we use to solve our problems. Few people stop to consider what it is they do when, for example, they fill in the stubs in their check books.

The check stub utilizes concepts of both number and numeral. What is a number? What is a numeral? There is a difference between them, and that difference is an important one for the reader who wishes to improve his understanding of elementary mathematics and the teaching of mathematics in the elementary school.

Let us think about the difference between the name of an object or thing and the thing itself. For example, when we think of the word "love," we are not necessarily experiencing love itself. We can discuss "ethics" without demonstrating ethical behavior. We can write the

words "Abraham Lincoln," but the words are not the man himself. We recognize that the name of the thing is not the thing itself. But suppose we write the word "two" or the symbol "2" on paper, on chalkboard, or in the sand. Is the collection of graphite particles on paper, the collection of chalk particles on the chalkboard, or the indentation in the sand the number two itself, or is it merely a name for the number—a symbol to call to mind through visual perception the thought of the number two? The name or symbol for a number is not the number itself. The idea of threeness is not inherent in the symbol or name for the number denoted "3" in our ordinary number system. The association or identification of the symbol "3" with the concept of threeness exists only in our own minds (see Figure 1-1).

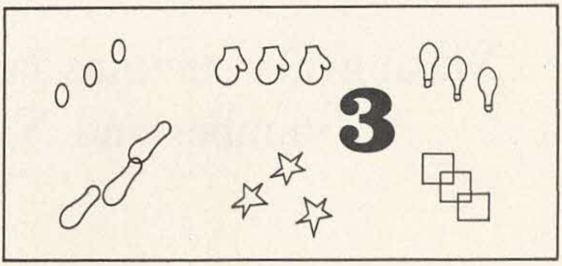

FIGURE 1-1

This idea can be further illustrated through consideration of our means of communication with those who are blind, deaf, and mute. It is common to communicate with such persons by the sense of touch. They learn, as do those who communicate with them, that when their hand or wrist is touched in a certain way a quantitative (number) relationship is being communicated to them; they learn to respond to such a stimulus without ever having had visual perception of the ordinary name or symbol for a number. Can the particular touch be a number, or is it a symbol which is used to stand for the number?

A numeral is a symbol which is used to stand for a number. If numerals are symbols, they can take many forms, as we all know. These symbols for numbers can take the form of spoken or written words in almost any language (see Figure 1-2), of written symbols (distinct from the written *word*) which may take many forms of hand signs, or of tactile sensations. We perceive these symbols through our senses of hearing, sight, touch. It would probably be possible to devise a system by which we could perceive symbols standing for numbers through our senses of smell and taste.

LANGUAGE	NUMBER NAME		
English	one	two	three
French	un	deux	trois
Spanish	uno	dos	tres
German	eins	zwei	drei

FIGURE 1–2

Why is it important that boys and girls understand these differences between numbers and numerals? Much misunderstanding results from a failure to recognize the distinction between number and numeral. Misunderstanding results when children say, "Three twos and two threes are the same." Are they the same?

On the abacus in Figure 1–3 we can see just what three twos look like. Two threes, however, look quite different. These can both be described with numerals and other arithmetical symbols. The former is described 3×2, whereas the latter is 2×3. The common element in the two situations is the fact that when each of these two or more groups is combined we have a total of 6 counters or 6 units.

Another source of confusion for the child is in the words we use in connection with numbers and numerals. When we ask, "How much is half of 8?" the child has memorized a response which tells him that half of 8 is 4. If the child is thinking about the number eight which

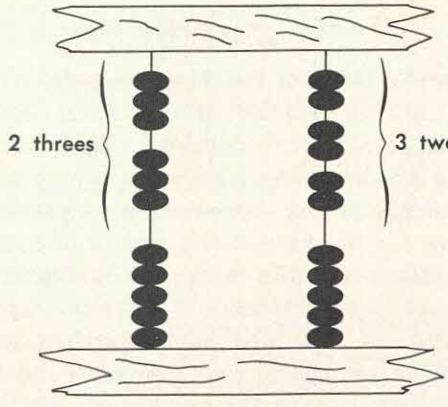

FIGURE 1–3

that symbol represents, his response of 4 is entirely correct. If however, he is thinking about the symbol "8," half could be any of a number of things. Half of the numeral 8 is zero (the top half)—or it is 3 (the right half). The same confusion can result from other numerical expressions we frequently use—for example, 21 take away 1 can leave 2 if we mean numeral, but something else if we are discussing numbers. Is 2.14 larger than 7? Of course it is, if we are discussing numerals rather than numbers.

How can we help children make the distinction between numbers and numerals? We are not talking about how we can get children to *say* there is a difference. The problem confronting the teacher is one of helping children understand that there *is* a difference, helping them understand *what* that difference is, and helping them *function* in the solution of problems as though there is a difference between numbers and numerals.

To recognize the difference we need a closer association between the numbers and the symbols or numerals we are using to describe these numbers. Children need many opportunities to relate the numerals they are using to the numbers these numerals describe. Many teaching aids can be used to assist in the achievement of this objective. As an example, let us turn to a problem in fractions: $\frac{4}{5} \div \frac{2}{5} = ?$ Number situations described by these symbols can be shown in two ways. First, with a circle cut into fractional parts as in Figure 1–4, how many $\frac{2}{5}$'s in a $\frac{4}{5}$? Second, with a total group of objects of which $\frac{4}{5}$ of the group is to be divided by $\frac{2}{5}$ as in Figure 1–5, how many two-fifths of the group are there in four-fifths of the group? The symbols "$\frac{4}{5} \div \frac{2}{5} = ?$" are understandable only in terms of the things they describe.

In developing an *understanding* of two-place multiplication, aids which provide examples of number situations can be used to show that the numerals used are describing what is happening in the rearrangement of number situations. Take for example the problem twelve times fourteen. In Figure 1–6, we have twelve groups of fourteen dots each. The question is: How many dots do we have altogether? We want to know how many are twelve fourteens. Twelve fourteens are two fourteens and ten fourteens. The first partial product in our algorithm describes the number of dots in two fourteens; the numerals in the second partial product describe the number of dots in ten fourteens.

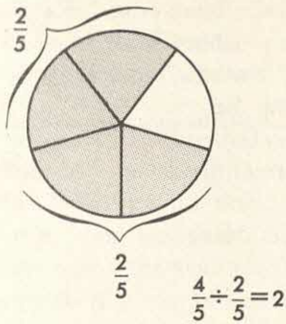

FIGURE 1–4

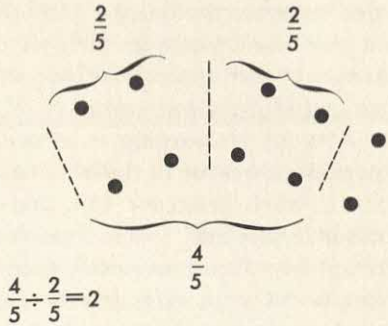

FIGURE 1–5

The final product uses numerals which describe the number of dots included in two fourteens and in ten fourteens.

Teachers also help pupils learn to make the distinction between number and numeral when they use appropriate language. We have already seen that some difficulties result when we ask children to perform certain operations with symbols, such as the case of taking one-half of 8, or 1 from 21. Teachers must make certain through their use of language that pupils understand that the 8 represents eight objects. They must make it clear that they want to know how many objects are in half of the eight objects, and that pupils can report their answer with a symbol

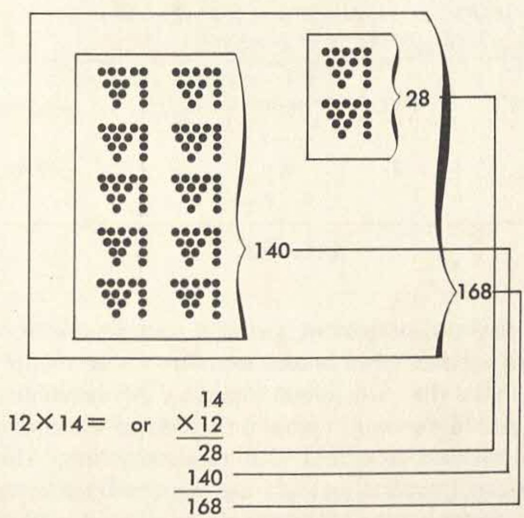

FIGURE 1–6

that describes the number of objects which would be one-half of eight objects. Arithmetic teachers in the elementary school need to make frequent and appropriate use of the terms: symbol, represents, describes, word name, and numeral, as well as number.

A word of warning is in order here. The teacher's insistence on pupils' correct use of these terms does not assure understanding of these terms. Such insistence can, and often does, result in confusion and misunderstanding. Instead, teachers should make frequent and appropriate use of correct terms, encouraging pupils to eventually incorporate both the meaning and the use of these terms as a part of their own mathematical competence.

Assuming, now, that we have a system of symbols to represent numbers—that is, a system of numerals—what can we say about these numerals? What properties do they have? Obviously, the properties possessed are dependent to a certain extent on the type of symbol which is used. If the symbols are words—alphabetic characters arranged in a particular sequence—then they have properties differing from those possessed by non-alphabetic symbols. Hereafter, we shall consider the "symbol" for a number to be different from the "name" for the number. The symbol will be thought of as being primarily non-alphabetic, and the name will be primarily a particular alphabetic sequence. Recognizing that this is an oversimplification of the problem, let us hereafter call the symbol the numeral and call the spoken or written word standing for the number its name (see Figure 1–7).

ONE	TWO	THREE	FOUR	—Names
1	2	3	4	—Numerals

FIGURE 1–7

Before leaving the subject of numerals, let us ask ourselves several questions. Are we inevitably bound to the symbols, the numerals which we now use? Are they a necessary part of numeration? Could other symbols be used just as well (from a theoretical standpoint as opposed to a practical one)? We would have to answer "no," that we are not bound to these numerals which we use, except from a practical standpoint. Other symbols could be used. In fact, any symbol could stand for any number. If a person were to use another set of numerals, there

would, of course, have to be common agreement about their meaning, but no theoretical difficulties would arise. That this is true can be seen merely by looking back just a little way in history. As late as A.D. 1600, Roman numerals were still being used by tradespeople in Europe for accounting purposes, and it was not until the invention and popular use of the printing press that the Roman system of numerals began to be displaced.

How can we teach that numerals are man-made? Numerals are used in communication. Children communicate with words and readily recognize that there is no one way to say a thing. Too often elementary school mathematics is taught as though there were only one correct way: one correct way to solve a problem; one correct way to write an answer; one correct way to describe a numerical situation. In one first-grade class, a child wrote on the board the numeral which told how many children were present in the room. He wrote 21. "Now put down the numeral which tells how many adults are now in this room," the teacher asked, and the child put a 3 under it. "And how many people do we have altogther?" The child wrote 24. The only thing wrong with this is that the child did not make the algorithm just as we would have made it. To insist that the child do this correctly at this time would not only put the emphasis on the wrong thing but would at the same time add to the illusion that there is only way of communicating in mathematics.

Another way to help children understand that numerals are man-made is to help them see that the numerals we use today have evolved through several centuries from forms both similar and not so similar to the forms of numerals we now use.

$$\boxed{\begin{array}{c} 21 \\ 3 \\ \hline 24 \end{array}}$$

The child further recognizes and understands this concept if he participates in the making of a few symbols which have meaning only to himself and his friends or classmates. While any marks can be used, let us construct for ourselves a series of numerals which only those of us reading these materials will understand. For one, let us use a horizontal line; for two, an angle; for three, a triangle; for four, a square; for five, a square with a line through it; for six, the symbol for five with a line under it; for seven, the symbol for five with an angle under it;

and so on. Now with these symbols, illustrated in Figure 1–8, we can describe the number of fingers we have on one hand; the number of eyes we have; the number of buttons on a cuff; and so forth.

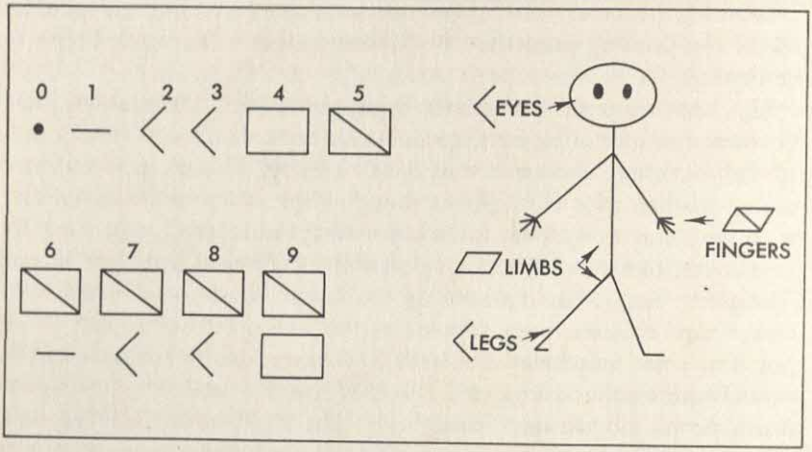

FIGURE 1–8

THE NATURE OF NUMBER

Finally, let us turn our attention to the subject of number itself. One of the issues of modern mathematics is the correct way to treat number. No mathematician would argue that numbers do not exist, but the manner in which they exist is the issue. Probably the most rigorous treatment of mathematics would contain one or more postulates concerning the existence of numbers (the natural numbers, whole numbers, counting numbers), but would leave the term "number" undefined.

We can examine "number" as a concept from several different vantage points. We can see what an examination of the historical development of the concept has to say to us, we can look at the psychological development of the number concept, and we can look at the number concept in the light of our own experiences and thoughts.

From an examination of the artifacts of primitive man, and from an examination of primitive cultures which still exist today, it seems relatively certain that man had some sense of number long before he was able to count or use numerals to stand for numbers. Historically, prim-

itive man seems to have had a sense of one-to-one correspondence of physical objects long before he was able to count. If he possessed more sheep than he could recognize at a glance, he seems to have used a pile of pebbles, or knots tied in a cord, or notches on a stick to denote how many sheep he had. He would use one pebble, let us say, for each sheep. At night, or at the end of the summer season, he could put the sheep into one-to-one correspondence with the pebbles in the pile. If pebbles were left over, he knew that *some* sheep were missing, or if sheep were left over, he knew that he had had an increase in the flock; but he was not yet in a position to tell how many. Later he learned to count.

This trick of putting objects into one-to-one correspondence without counting is not just an ancient device. Many modern examples of it are to be found. In keeping score in some games, the winner is not determined by the total number of marks but by having accumulated more marks than his opponent. During World War II, many men and women who were imprisoned made marks on the walls of their cells to correspond to the days during which they occupied their cells. No special thought of counting was involved, but merely a putting into one-to-one correspondence the days and the marks. Later, possibly, these marks were interpreted by others. Long after man began to put objects into one-to-one correspondence (a very basic procedure for intelligent beings), he apparently began to group his pebbles, or knots, or notches into groups of uniform size. These groups might have been groups of two (for the number of wings which a bird always was observed to possess) or groups of five (like the number of fingers on the hand). When he began to group, he was beginning to see the abstraction of number from object. Whether he had five sheep, five fowl, or five members of his family, he was beginning to see that there was a "fiveness" about the objects which was distinct from the objects. He might still keep track of the objects by a one-to-one correspondence between a group of five pebbles and the group of objects being accounted for, but somewhere and sometime he began to realize that the pebbles in the pile were not the objects being accounted for. He recognized that there was some intrinsic property of the set of pebbles which was also an intrinsic property of the set of members of his family, or the set of sheep in his flock, or in the set of fowl which he had caught. This was the beginning of number concept. Many primitive peoples never got beyond the recognition of "oneness," "twoness," and "manyness." Some primitives apparently never made the abstraction at all, for to some primitives two fish were entirely different from two coconuts and had no property in common. In fact, until the date

of their extinction or absorption into other tribes and peoples, some primitives always used different word sounds to denote two fish, two coconuts, two pigs, two boys, etc.

In conclusion, we should mention some other facets or areas which could be investigated in a consideration of numbers and numerals. The study of numerology—of the mystical properties of numbers and numerals—while not scientific, is entertaining. We could also study systems of numeration. Or, finally, we might consider the problem of what our numeral system would have been like if, since the beginning of time, we had been a race of people who had six fingers on each hand rather than five.

EXERCISES

1. In the following phrases the expressions enclosed in quotations represent the symbolic aspect of number, while the expressions not enclosed by quotations represent the conceptual aspect of number. Which of the following statements is incorrect? Explain. Expand this list to include others.
   (a) "3 + 4" is a numeral for 7.
   (b) you can put "7" on the chalkboard.
   (c) "7" is an odd number.
   (d) 9 is divisible by 3.
   (e) 7 is a numeral for "3 + 4."
   (f) "3" + "4" is a numeral for 7.
   (g) there are 8 "4's" in 32.
2. Insert quotation marks in each of the following sentences wherever necessary to denote numerals:
   (a) Add 5 and 4.
   (b) 15 is a symbol for the number 5 + 10.
   (c) The 5 you wrote on the chalkboard is poorly formed.
   (d) You can subtract 5 from 12, but you cannot subtract 5 from 12.
3. When you hold up four fingers to show that there are four persons in a group, are the four fingers a number, a numeral, or neither? If the fingers signify four fingers, does this fact change your answer to the first question?
4. Relate your answer to Exercise 3 to one-to-one correspondence.
5. An usher in a church holds up two fingers to signal an approaching couple. What significance may his upraised fingers have? Do they constitute a number or a numeral? To what sets may they be put into one-to-one correspondence?

6. How many numerals are there which represent the number eight? Discuss.

7. Write a set of four examples like those in Exercises 1 and 2.

8. To find if there is a one-to-one correspondence between the legs of a chair and the fingers of your hand, you check and find that for each chair leg you have a finger. Does this establish a one-to-one correspondence? Discuss.

9. Devise a system of numerals which depends on the sense of touch and demonstrate it to your class.

10. (a) What would be the most primitive way to show or symbolize the number of a set of 100 coconuts?
    (b) What would be the next step in the direction of abstraction in symbolizing the 100 coconuts?
    (c) Describe several more abstract levels of symbolizing the number of the set.

11. (a) Do we use symbols for non-mathematical ideas, objects, or operations? Name some.
    (b) Are these symbols the same as the objects represented?

12. Give an example of two sets which do not have the same (cardinal) number of members. Show this by matching, not by counting.

13. Investigate and report on finger numerals and how to use them.

14. Discuss the merits of the phrase "two-digit number." Does the title, "Addition of two-place numbers," used in some texts, violate the meaning of number? Do you think that the title, "Addition of two-place numerals," is better? Why?

15. Make a set of numerals and use your symbols in writing and solving the following problems.

$$\begin{array}{r} 43 \\ +26 \\ \hline \end{array} \qquad \begin{array}{r} 24 \\ +38 \\ \hline \end{array} \qquad \begin{array}{r} 33 \\ -18 \\ \hline \end{array} \qquad \begin{array}{r} 123 \\ -\ 47 \\ \hline \end{array} \qquad \begin{array}{r} 26 \\ \times 35 \\ \hline \end{array}$$

16. Trace the development of Hindu-Arabic numerals. What evidence can you find that these symbols are currently in process of change?

Extended Activities

1. Quantitative terms are in constant use today. Some are definite, such as 234 or 14 per cent, while others are indefinite, such as "more," "most," "longer," "fuller," etc. Circle all the quantitative terms on one page of a

daily newspaper. How many were there in all? How many were indefinite quantitative terms? How many were definite terms?

2. Investigate the use of the knotted cord as a device to show number.

3. Investigate the use of the notched stick as a device to show number. What terms are in common use today as a result of the use of the notched stick? (stock, stockholder, the story of Parliament burning down, etc.)

4. Trace the development of the abacus as a recording and computing device. Explain the differences found in the Chinese, Japanese, and Persian abaci and the Roman counting board. What words derive their use and being from the abacus? (calculate, bank, bankrupt, carry, borrow, counter)

5. Read and report on the book *Infinity* by Lillian R. and Hugh Gray Leiber (New York: Holt, Rinehart & Winston, Inc., 1953).

6. Read and report on the article "Definition of Number" in *The World of Mathematics* by James R. Newman (New York: Simon and Schuster, Inc., 1956).

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Chapter 2

Concepts to be developed in this chapter are:

1. *The ideas of "set" and "matching of sets" are basic concepts out of which our ideas about number have grown.*
2. *"Set" is the term mathematicians use when dealing with collections of objects, things, symbols, or ideas.*
3. *A set may be described in terms of its component parts which are called elements.*
4. *The set concept may be considered useful in clarifying and redefining established mathematical relationships for teachers and children.*
5. *Symbols and systems of notation currently being developed to designate sets, their elements, and the procedures used in working mathematically in terms of sets must be adequately defined in order to achieve communication.*
6. *The Venn (or Euler) diagram is useful to show graphically the relationship between sets and elements of sets.*
7. *Each of the four fundamental processes may be defined in set terms.*

2

Exploring the Use of Sets in Elementary Mathematics

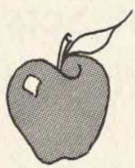
Regarded by many as the basis for the number idea, the set concept is a potent tool, one with which elementary school teachers of mathematics should be familiar. In this chapter we shall consider a few of its basic meanings, components, and possible uses in the elementary school.

THE MEANING OF SETS

What is a set? Some authorities refuse to define it, declaring it to be a concept so basic that it is undefinable except by synonym. Two common synonyms are "class" and "collection." For our purposes, we shall describe a set as a well-defined collection of objects, things, or symbols which, by mental association, is regarded as a whole. The items regarded as a part of a set will be referred to as "elements" or "members" of the set.

In and out of the classroom, children and adults see and deal with sets of elements daily, as the following examples illustrate: classes of children, sets of dishes, sets of jewelry, stacks of books, cartons of drinks, packs of cigarettes, sets of flash cards, sets of minutes, sets of coins. Sets are frequently composed of elements which are related logically or physically, as are the elements of the above mentioned sets. Because elements are members of a set by definition, however, they *may be* concrete or abstract, homogeneous or heterogeneous, or any combination of these. Let us consider, for example, the following sets and their elements.

Set *A:* A collection of three elements not ordinarily related
Elements:



Apple

Father Time

Paris

Set *B:* Three things in my desk drawer
Elements: (a) Green crayon
(b) Ream of paper
(c) Snapshot of a friend

Set *C:* Three "spiritual gifts"
Elements: (a) Faith
(b) Hope
(c) Charity

Set *D:* All brothers in a family
Elements: (a) Eeny
(b) Meeny
(c) Miney
(d) Archibald

Set *E:* Ordered pairs of natural numbers with sum 4
Elements: (a) 1, 3
(b) 2, 2
(c) 3, 1

Set *F:* The set of reasons that my leg is in a cast
Elements: (a) A dark night
(b) A burned-out light bulb
(c) Haste

(d) A toy on the floor at the head of the stairs
(e) A fall
(f) A broken tibia
(g) A trip to the doctor

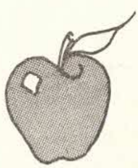
Set *G:* Natural numbers and zero
Elements: (a) 0
(b) 1
(c) 2
(d) 3 . . .

Set *H:* All living men on Earth over twelve feet tall

What is it that ties the diverse kinds of elements and sets together and gives them mathematical significance? These sets (including the infinite set *G*) and all others have one constant common property: they each serve as a basis for assignment of number. To clarify this statement, let us reconsider sets *A*, *B*, and *C*.

Although their respective elements lack the common properties of concreteness, color, usage, etc., they do have one mutually identifying property. As shown below, each element of *A* may be matched with one element of *B;* each element of *B* with one element of *C;* and (thus) each element of *A* with one element of *C*.

Set *A:*



Set *B:*	Green Crayon	Ream of paper	Snapshot of a friend
Set *C:*	Faith	Hope	Charity

We may say, then, that sets *A*, *B*, and *C* have the same number of elements, or that the number of set *A* = the number of set *B* = the number of set *C*.

Further, we may identify any set whose elements may be placed in exact one-to-one correspondence with those of set *A* as its numerical

equivalent; we speak of such sets as *equivalent* sets. We can also, by the same matching process, identify the number of any other set as being greater than, or less than, the number of set A.

Man first observed sets of objects. Later he found that he could represent these sets of objects from his environment by placing them into one-to-one correspondence with portable objects, such as stones, shells, etc. This portable set might be called a model set. Still later, he found that he could represent this set by numerals.

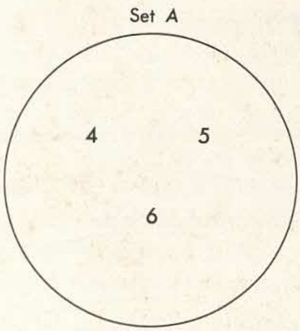
Many authorities feel that the set concept is important, not only historically, but currently, to facilitate mathematical pioneering, and to clarify and redefine established mathematical relationships. The concepts, symbols, and systems of set notation are being experimented with at all educational levels. They are being used in advanced mathematics, in high school algebra, geometry, and trigonometry, and in junior high school and upper-elementary-grade arithmetic. In time, the definitions of the fundamental processes in set terms may well affect mathematical activities and methods in all elementary grades. In other areas, as well as in mathematics, set concepts, notation, and diagrammatic representations may be used by teachers and children to clarify relationships.

BASIC SET TERMINOLOGY

To establish further communication regarding sets, we must define some of the basic set terminology. In our explorations we shall make frequent use of Venn diagrams, which graphically show set relationships, as well as of set notation and symbols. Some of the illustrative examples will be taken from daily life; some might be used by children; others might be useful to teachers in dealing with children. The use of set terminology with children should be consonant with their general and mathematical maturity.

In set notation, a set is designated by a capital letter, and its elements, when designated by letters, are represented by small letters. Set elements are enclosed in braces and may either be listed individually or identified by a descriptive phrase. The set A of all natural numbers between 3 and 7, then, could also be written: $A = \{4, 5, 6\}$. A Venn diagram of set A might appear as follows.

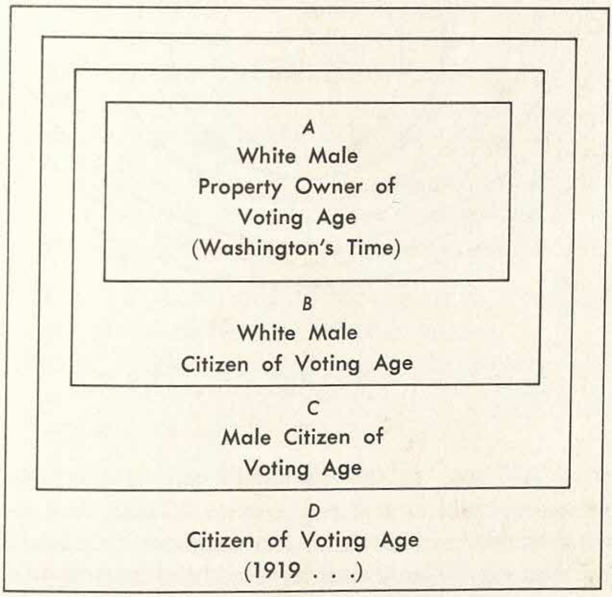
If all the elements of a set A are also elements of a set B, then A is termed a *subset* of B. This relationship may be written $A \subseteq B$ *or* $B \supseteq A$. The statement $A \subseteq B$ may be read either as "A is a subset of B" or as



"A is included in B." $B \supseteq A$ may be read as "A is a subset of B" or as "B includes A." One special case is worthy of note: If all the elements of A are also elements of B, *and* if B has at least one element not in A, then A is termed a *proper subset* of B. The statement "A is a proper subset of B" is symbolized $A \subset B$.

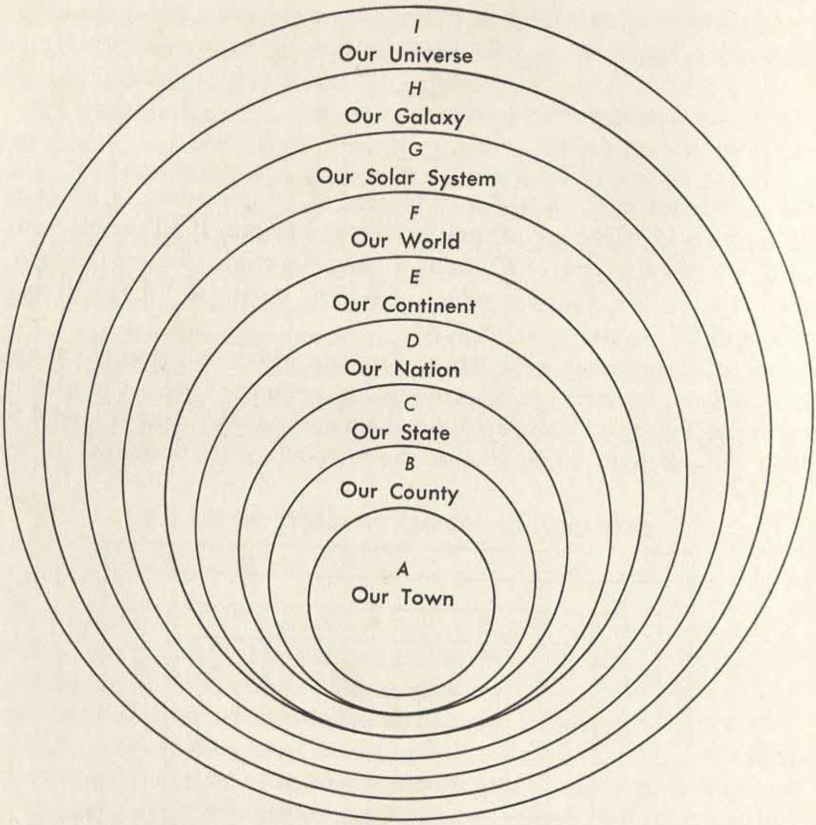
We deal constantly with sets and subsets. When we send clothes to the cleaners, for example, we are sending a subset of the set of clothes we own. The two following Venn diagrams, which might be useful in intermediate grade social studies, show set-subset relationships.

EXTENSION OF VOTING PRIVILEGES IN THE U.S.



$A \subset B \subset C \subset D$

WHERE IS OUR TOWN?



$A \subset B \subset C \subset D \subset E \subset F \subset G \subset H \subset I$

The terms "set" and "subset" should be easily understood by many elementary school children and might be used frequently in describing and directing various mathematical learning activities. For instance, in exploring the set of 100 basic addition facts, children might be asked to find the subset whose elements equal 7 (that is, $0 + 7$, $1 + 6$, $2 + 5$,

$3 + 4$, $4 + 3$, $5 + 2$, $6 + 1$, $7 + 0$) or the set of addition-subtraction or multiplication-division facts of a "number family" (for example, $3 + 4$, $4 + 3$, $7 - 4$, $7 - 3$) (or, 2×4, 7×8, $56 \div 7$, $56 \div 8$). In studying facts, children often work with individual sets of flash cards, segregating a subset of facts on which they need further study.

All multiple choice exercises, such as might be devised for estimating answers or identifying the correct process to use in solving a problem, involve a set of answers from which the student is to choose the correct element or elements. The following are examples.

1. Circle the correct element of the following set of estimated answers.

   $.24\overline{)49}$ (a) about 2, (b) about 20, (c) about 200
2. Circle the element which describes the process to be used.

   Mary ordered some Christmas cards to sell. She is to sell each box for \$1.50. She may keep 40 per cent of the selling price as her profit. How much profit will she make on each box? (a) addition, (b) multiplication, (c) division.

Suppose that we are invited to a friend's house and he offers us the following refreshments: coffee, cake, and candy. The set of all refreshments being offered might be called $A = \{\text{coffee, cake, candy}\}$. What could we choose to eat and drink from the refreshments being offered? In other words, what are all the possible subsets of A? If we were hungry, we might choose set B: $B = \{\text{coffee, cake, candy}\}$. If a subset has *all* the elements of set A, as does set B, it may be referred to as an *improper subset.*

We might also choose any of the following subsets of A.

$C = \{\text{coffee}\}$ $\quad$ $F = \{\text{coffee, cake}\}$
$D = \{\text{cake}\}$ $\quad$ $G = \{\text{coffee, candy}\}$
$E = \{\text{candy}\}$ $\quad$ $H = \{\text{cake, candy}\}$

Sets C through H may be termed *proper subsets* of A because, in each case, at least one of the elements of set A is missing.

Suppose that we decided not to have any refreshments. The set represented by this choice is called the *empty* or *null* set and is considered a subset of every set. It is usually denoted by either

$$\Phi, \epsilon, \text{or } \{\quad\}.$$

Set A is itself a subset of a more general set composed of all food and drink. A general set containing all elements fitting a certain description is usually referred to as the *Universe*, and in this case would be denoted as $U = \{\text{all food and drink}\}$.

UNION, INTERSECTION, AND EQUIVALENCE OF SETS

Set union, intersection, and equivalence are defined in terms of their elements. Two sets are said to be *disjoint* when they have no elements in common. Set A: {New York, Chicago, Los Angeles} and Set B: {Philadelphia, Boston, Dallas} are disjoint sets. The union of sets A and B (written $A \cup B$) may be shown as follows: $A \cup B$ = {New York, Chicago, Los Angeles, Philadelphia, Boston, Dallas}.

Two sets are said to be intersecting if they have some or all of their elements in common. The *intersection* of sets (written $A \cap B$) consists of those elements and only those elements common to both sets. Thus, given two intersecting sets: A = {New York, Chicago, Dallas} and B = {Dallas, Philadelphia, Boston}, $A \cup B$ = {New York, Chicago, Dallas, Philadelphia, Boston}, and $A \cap B$ = {Dallas}.

Two intersecting sets are said to be *identical* if each is a subset of the other ($A \subseteq B$ and $B \subseteq A$); that is, if all elements are common to both sets. If sets A and B are identical, we say $A = B$. We do not say $A = B$ when the sets are merely equivalent. Thus, Set A: {New York, Los Angeles, Chicago} = Set B: {The three largest cities in the United States}.

The *complement* of a set A (written $\widetilde{A}$) consists of those elements of the Universe which are not in set A. If, therefore, set A = {New York, Chicago, Los Angeles} and U = {New York, Chicago, Los Angeles, Philadelphia, Boston, Dallas} then $\widetilde{A}$ = {Philadelphia, Boston, Dallas}.

To further illustrate the meaning of these terms, we shall use Venn diagrams, a hypothetical situation, and the following sets. Each of these four sets contains a family composed of father, mother, and offspring.

A = {John Smith, *Mary Jones Smith*, Bill Smith, Susan Smith}
B = {Jack Dinklehoff, Kay Dinklehoff, Sam Dinklehoff, Ann Dinklehoff}
C = {Joe Jones, Eva Jones, *Mary Jones Smith*, Bob Jones}
D = {The father of Bill and Susan Smith, the mother of Bill and Susan Smith, the son of John and Mary Smith, the daughter of John and Mary Smith}

Each evening after supper, set A goes for a drive in their new station wagon. On three consecutive evenings, set A was joined by sets B, C, and D, respectively, for the drive. How can we diagram these set unions?

Sets A and B are disjoint, since they have no common elements.

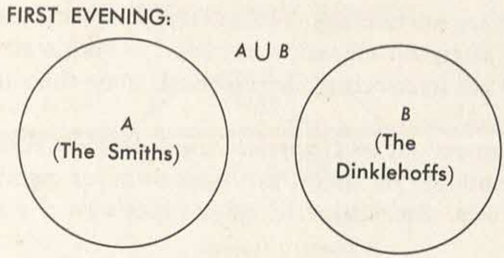


FIGURE 2–1

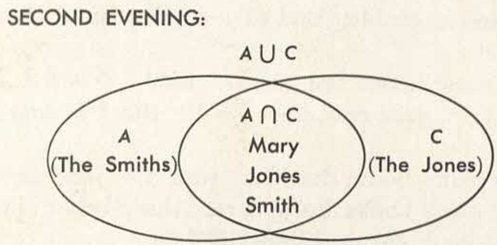


FIGURE 2–2

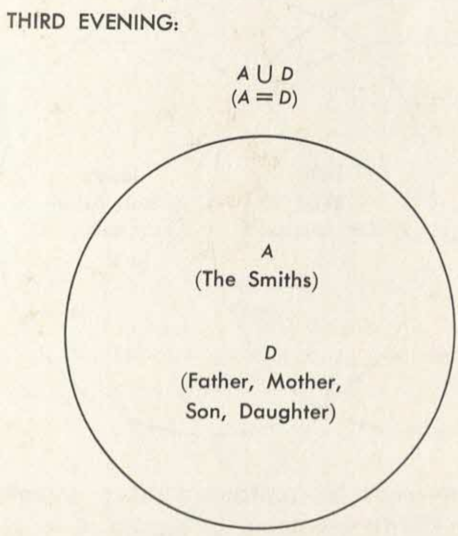


FIGURE 2–3

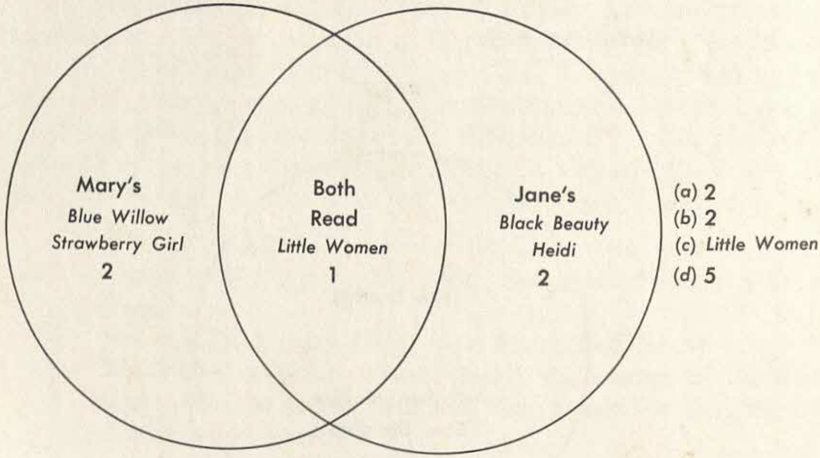
Sets *A* and *C* are intersecting. The intersection is the element "Mary Jones Smith," since that element is common to both sets.
Sets *A* and *D* are intersecting and identical, since their membership is identical.

Both children and adults frequently meet problem situations involving set relationships. At times, the assignment of number to sets is desired; at others, clarification of other aspects of the relationship is sought.

Without great emphasis on formal definition of terms, these concepts are being introduced to some upper-elementary-grade children through Venn diagrams. The diagrams serve as models of the problem situation, as we can see in the problem and solution diagrams below.

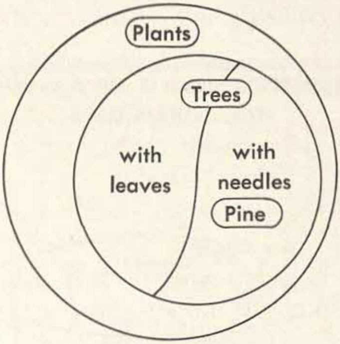
Mary read these books last month: *Little Women*, *Blue Willow*, *Strawberry Girl*. Jane read these books: *Black Beauty*, *Heidi*, *Little Women*.

(a) How many books did Mary read that Jane didn't read?
(b) How many books did Jane read that Mary didn't read?
(c) What book did they both read?
(d) How many different books were read by the two girls?



Simple diagrams might be constructed and/or intepreted by upper grade children to clarify non-numerical aspects of set relationships.

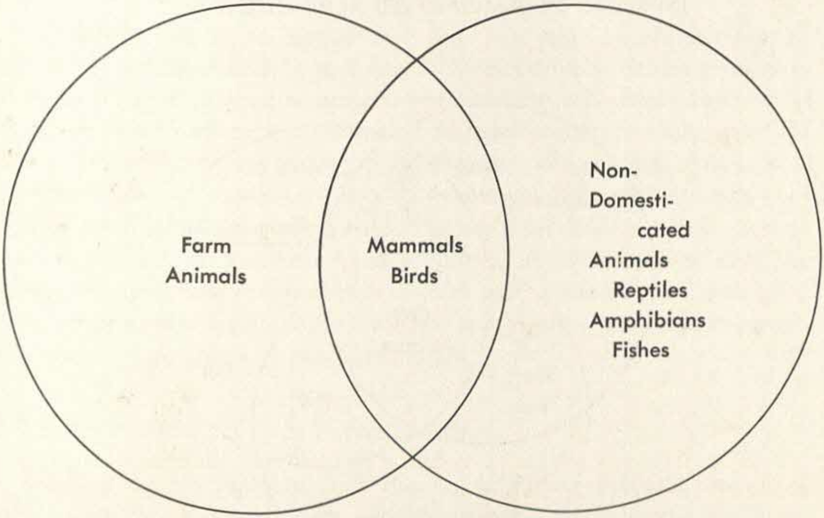
CLASSIFICATION OF PINE TREE



VERTEBRATES:

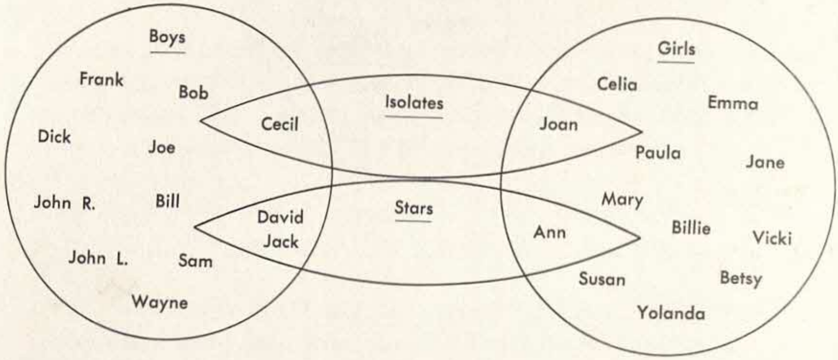
FARM AND

NON-DOMESTICATED ANIMALS

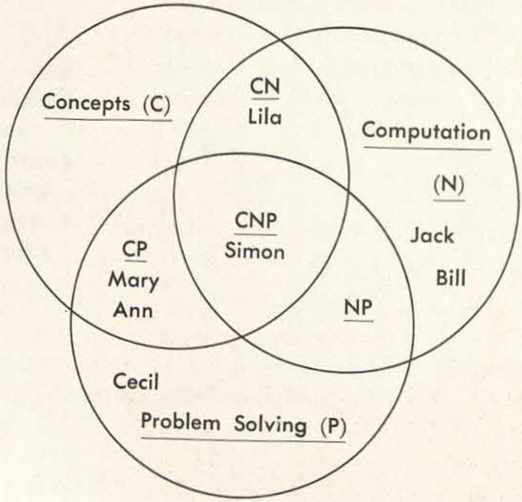


The following diagrams illustrate models for the solution of common problems encountered by teachers in getting to know and in evaluating the progress of their children.

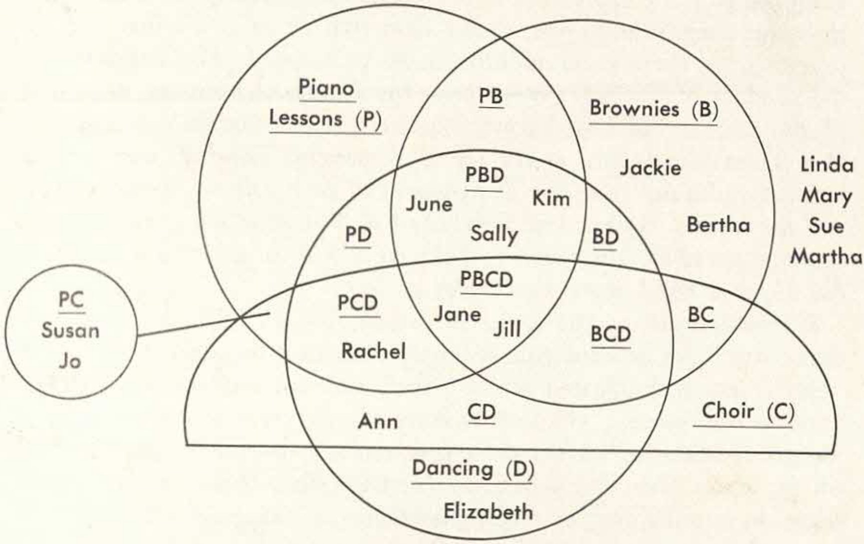
SOCIOMETRICS: RESULTS OF ANALYSIS OF SOCIOGRAM DATA



CHILDREN BELOW GRADE LEVEL IN ARITHMETIC CONCEPTS, COMPUTATION, AND PROBLEM SOLVING AS INDICATED BY STANDARDIZED ACHIEVEMENT TEST



ANALYSIS OF OUTSIDE ACTIVITIES OF GIRLS IN CLASS



SETS AND ARITHMETIC

Addition Defined in Set Terms

Let us now return to Figures 2–1, 2–2, 2–3, page 23, and attempt to add sets A and B, A and C, and A and D, respectively, defining addition only as being a process associated with uniting sets. Since each set is made up of four elements, a union of any two should produce a sum of eight. However, in reviewing these Figures we see that only $A \cup B$ creates a sum of eight ($A \cup C$'s sum being seven, and $A \cup D$'s sum being four). It becomes clear, then, that we must limit our definition of addition of natural numbers to include only *disjoint sets*, and that *the number assigned the union of two disjoint sets, A and B, must match the number of a third set C whose members are placed in one-to-one correspondence with those of the united sets.*

Multiplication as a Set Operation

Most of us are familiar with the definition of multiplication as a process of successive additions. Although adequate in many instances, particularly during the early elementary school years, this definition necessitates the formulation of additional definitions to cover special

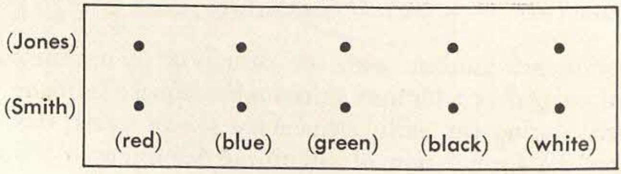
cases. For example, although we can rationalize "7×1" as 7 successive additions of 1, the expression "1×7" (one times seven) has little meaning since there is only one, rather than two or more 7's involved; in other words, there is no addition to be performed. The definition of the product of two sets gives a basis for a more satisfactory definition of the product of two natural numbers. *The product of two sets, $A \times B$, may be defined as the set of all possible ordered pairs formed from A and B such that the first element of each pair is a member of A, and the second element is a member of B. The product of two natural numbers a and b is the number of the set $A \times B$, in which a is the number of set A and b is the number of set B.*

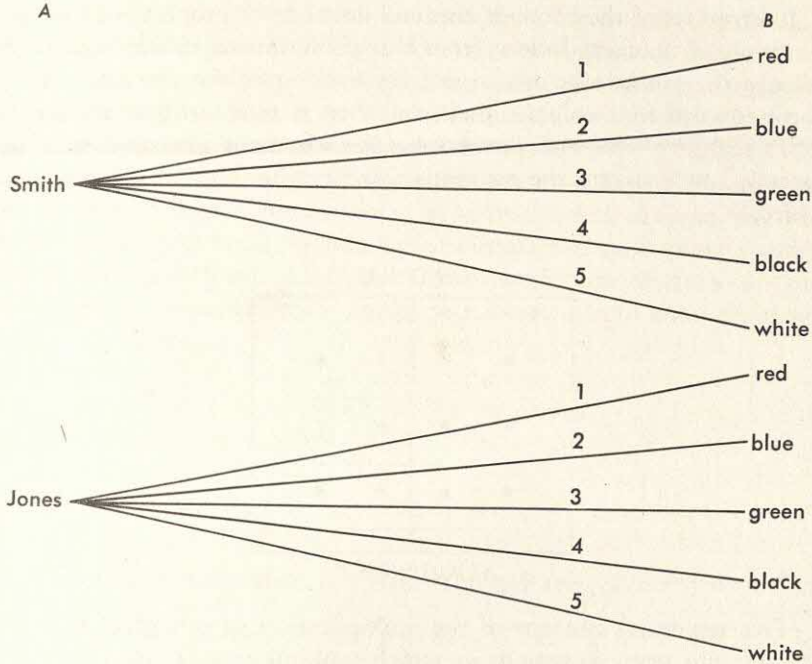
Before considering the entire definition, let us clarify the term "ordered pair." An ordered pair is simply a set of two elements in which one element is designated as the "first" element, and the other designated as the "second" element. In daily life we often deal with ordered pairs. For example, we (a) *dress* first, and (b) *go out* second; we put on (a) socks, then (b) shoes; we (a) use dishes before we (b) wash them. In constructing or reading a height-weight graph, the order of numerals is significant: 105 in., 62 lb. does *not* represent the same body dimensions as 62 in., 105 lb., nor is it so represented on the graph. On such a graph, in fact, there is one and only one ordered pair (set of coordinates) which corresponds to each point on the graph, and vice versa.

Suppose that we have the following two sets: $A = \{\text{Men}\} = \{\text{Smith, Jones}\}$, $B = \{\text{Automobiles}\} = \{\text{red, blue, green, black, white}\}$. The five automobiles are on an assembly line. Each of the two men works on each automobile as it comes by. According to our definition, to determine the number of the product of these sets, we must find how many different pairings of men and automobiles are possible. One way of representing the product is by a tree diagram (see p. 29).

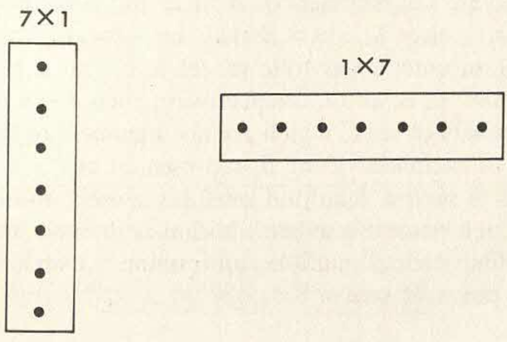
The set of all ordered pairs is illustrated, and we see that the number of the set of all possible ordered pairs from sets A and B is ten. Another method of determining the number of ordered pairs is by associating the product with a rectangular pattern, each point of which represents one pair, as follows.

$A \times B$

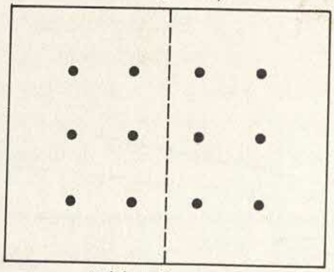




Any product can be thus represented by a rectangular set of points. Omitting a more technical description of this pattern, we can say that each point of the pattern represents membership in both a row and a column. If we wish, then, to represent both "7×1" and "1×7" we can do so logically: the first pattern would represent the possible pairings of members of a set A containing seven members with that of a set B containing one member; the second would represent the set of pairs which could be formed from a set A containing one member and a set B containing seven members.



Illustrations of the commutative and distributive properties of multiplication of numbers follow from this definition of multiplication of sets. In the example below, by merely looking at the pattern, first as three rows of four columns each, and then as four rows of three columns each, we "see" that $3 \times 4 = 4 \times 3$. (The same principles may be extended to illustrate the associative property if a third dimension is added.)



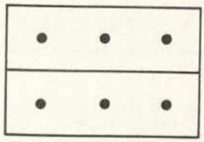
This structural concept of the multiplication of sets gives us fresh insight into many situations in which multiplication is used. For example, in finding area by the formula $A = LW$, we can see that each square unit is the result of a pairing of length and width units, and that the number assigned as the area is the number of the set of all possible pairs so formed.

Partitioning of Sets

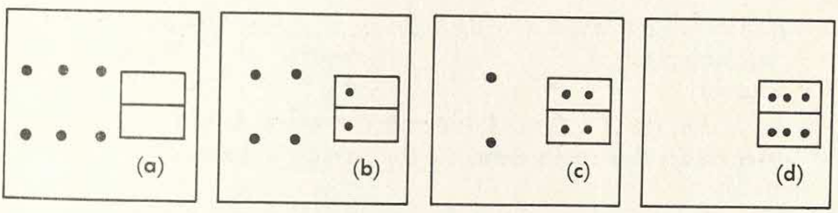
Set partitions may be accomplished by subtraction and division, which are inverse operations of union and multiplication, respectively.

Subtraction of sets. For sets A, B, and C, subtraction is defined as follows: $C - A = B$ if and only if B consists of precisely those members of C which are not members of A. If A and B are disjoint sets for which $A \cup B = C$, then $C - A = B$ may be associated with ordinary natural number subtraction as follows: let a, b, and c be the number of elements in sets A, B, and C, respectively; then $c - a$ represents the number of members of set C which are not members of set A, and this is the number of members of set B, represented as $c - a = b$. Because the restrictions in such a definition resemble closely those imposed on the sets for which natural number addition is defined, it is somewhat simpler to define natural number subtraction in terms of addition rather than in terms of sets.

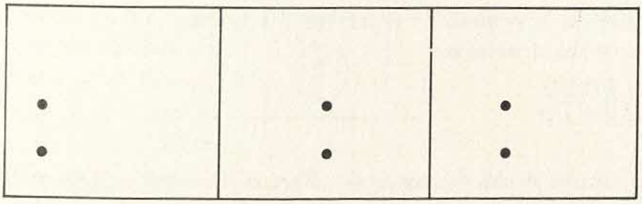
Division of sets. If $A \times B = C$, then $C \div A = B$ and $C \div B = A$. With such a definition of division for sets, a definition of division for natural numbers could be fashioned, but it is simpler to define division of natural numbers in terms of multiplication of natural numbers (which will be done in a subsequent chapter). However, we can view a set as being partitioned, or divided, in two ways. Given the set and the number of subsets into which it is to be divided, division may be performed to determine the number of elements in each subset (partition division situation). Thus, the equal distribution of six sheets of paper between two children would be associated with the following rectangular pattern.



This could be accomplished intuitively through the following sequence.



The following pattern describes the other division situation, in which, given the set and the number of elements in each subset, division is performed to determine the number of equal subsets (measurement division situation). Here, we determine how many children can be supplied with two sheets of paper each from a set of six sheets.



Although representative of two distinct division situations, all division by the standardized division process is accomplished by the measurement procedure; that is, by repeated subtractions of *equal subsets*

(the divisor), the *number of subsets* (quotient) contained in the dividend (set) is determined.

EXERCISES

1. Establish a one-to-one correspondence between the members of $A = \{a,b,c,d,e\}$ and $B = \{1,2,3,4,5\}$.

2. In Exercise 1, in how many ways can the one-to-one correspondence be established?

3. $A = \{a,b,c,d,e,f,g\}$
$B = \{b,d,e,h,j,k,a\}$
Find $A \cap B$, $A \cup B$, and $(A \cup B) - (A \cap B)$.

4. If $A = \{a,b,c,d,e,f\}$, $B = \{c,d,e,f,g,h,i,j\}$, $U = \{\text{the alphabet}\}$, and $C = \{x,y,z,w,v,u,t,s\}$.
what is:
(a) $A \cup B$; (b) $A \cup C$; (c) $B \cap C$; (d) $C \cap U$; (e) $C \cap A$?
(Remember that A represents the complement of set A.)

5. Draw Venn diagrams and shade appropriate areas to illustrate:
(a) $\widetilde{A} \cap B$ (given that $A \cap B \neq \Phi$);
(b) $(\widetilde{A} \cup \widetilde{B}) \cap C$ (given that $A \cap B \neq \Phi$, $A \cap C \neq \Phi$, $B \cap C \neq \Phi$).

6. Draw a Venn diagram which will illustrate:
$A \cup B = U$

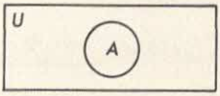
7. Let $n(A)$ be a symbol to represent the number of elements in set A. For Exercise 4 what is
(a) $n(A \cap B)$?
(b) $n(A \cup \widetilde{C})$?

8. For Exercise 4 show that $n(A \cup B) < n(A) + n(B)$. (The symbol $<$ is read "is less than.")

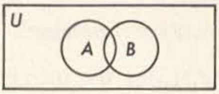
9. If $n(A)$ represents the number of set A, show that $n(A) + n(B) \geqq n(A \cup B)$. (The symbol $\geqq$ denotes "is greater than or equal to").

10. Shade the appropriate areas of the accompanying Venn diagram to illustrate the specified sets in the following problems.

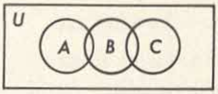
(a) $\widetilde{A}$



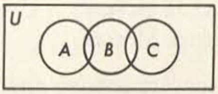
(b) $\widetilde{A} \cup B$



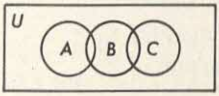
(c) $(\widetilde{A \cup B}) \cup C$



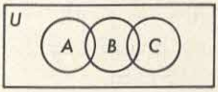
(d) $(A \cap B) \cup C$



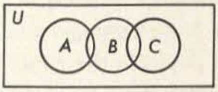
(e) $\widetilde{A} \cap \widetilde{C}$



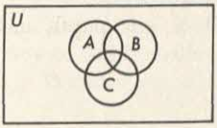
(f) $A \cup C$



(g) $\widetilde{A} \cup \widetilde{C}$



(h) $\widetilde{A} - (B \cup C)$



11. The general definition of the subtraction of sets does not correspond exactly with the definition of subtraction for natural numbers. Use the set definition to find the following differences given that $A = \{1,2,3,4,5\}$ $B = \{4,5\}$ $C = \{4,5,6\}$ $D = \{1,2,3,4,5,6\}$:

(a) $A - B$ (b) $A - C$ (c) $A - D$ (d) $B - D$ (e) $D - A$

12. Using the sets defined in Exercise 11, find these numbers:
(a) $n(A-B)$
(b) $n(A-C)$
(c) $n(A-D)$
(d) $n(B-D)$

13. Using the sets defined in Exercise 11, find these numbers:
(a) $n(A)-n(B)$
(b) $n(A)-n(C)$

14. Is it true that $n(X-Y)=n(C)-n(Y)$? Explain.

15. (a) If $A=\{a,b,c\}$ and $B=\{p,q,r,s\}$, list the membership of $A \times B$.
(b) Is $n(A \times B)=n(A) \times n(B)$?

16. Write the members of the following sets.
(a) the set of integers from ten to fifteen.
(b) the members of your immediate family.
(c) the vowels in the alphabet.
(d) the odd numbers less than twenty.
(e) the cars made by General Motors.

17. Which of the following sets are infinite sets?
(a) the citizens of the United States.
(b) the atoms in our world.
(c) the natural numbers evenly divisible by 7.
(d) the fractions with a denominator of 2.

18. Which of the following are empty sets?
(a) men who can jump 7 feet high.
(b) odd numbers which are exactly divisible by 2.
(c) squares of odd numbers that are even.

19. Make a Venn diagram to illustrate the following situations.
(a) the pupils in your room; the boys in your room.
(b) the pupils in your room; the pupils in the room next door.
(c) the set of natural numbers from 1 to 10; the set of odd numbers; the set of numbers divisible by 3.
(d) the pupils in high school; the pupils named Mary; the pupils 16 years old.

Extended Activities

1. Draw one Venn diagram that shows the relationships between all of the following sets:

$A=\{\text{all real numbers}\}$
$B=\{\text{all natural numbers}\}$

$C = \{\text{all negative numbers}\}$
$D = \{\text{all rational numbers}\}$
$E = \{\text{zero}\}$

2. (a) List all the subsets (ignoring the order of the elements) of the set $\{a,b,c,d\}$. Be sure to include the empty set. Count the number of subsets. Do the same for the set $\{a,b,c\}$.
   (b) Try to conclude how many subsets a set has.
   (c) Check your conclusion by listing and counting all the subsets of $\{a,b,c,d,e\}$.

3. Given: (1) $A \cup A = A$
   (2) $A \cap U = A$
   (3) $A \cup \widetilde{A} = U$
   (4) $\widetilde{A \cup B} = \widetilde{A} \cap \widetilde{B}$
   (5) $A \cup (B \cup C) = (A \cap B) \cup (A \cap C)$

   Prove: $A = (A \cap B) \cup (A \cap \widetilde{B})$.
   Justify each step of the proof.

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Chapter 3

Concepts to be developed in this chapter are:

1. *The communication of quantitative relationships requires not only symbols but also systems of arranging these symbols.*
2. *The base we use is the one man has selected for use. Any of several others could have been selected and used with the same effectiveness.*
3. *In everyday activities we sometimes use other bases. The most frequently used bases other than ten are twelve and sixty.*

3

Identifying Characteristics of Base and Place

THE NATURE OF POSITIONAL NOTATION

The abacus, place value chart, money, and peg boards are all effective aids used by teachers in helping children develop concepts of base and place. We will use the peg board to illustrate the development. The crosspiece on the peg board shown in Figures 3–1 through 3–5 is placed in the tens position. As we count pegs, we put them in holes which are placed in a straight line from top to bottom. When we count the tenth peg, we find we have no hole to put it into (Figure 3–1). Instead we gather all ten(the nine in the holes and the one in our hand) into a bundle and put them aside.

Before we count any more pegs, let us stop and think for a minute. There are two ways we can tell that there is now a bundle of ten pegs. First, we can see by its size that it is a bundle of more than one peg (and we will have to be willing to agree that the pegs were not gathered

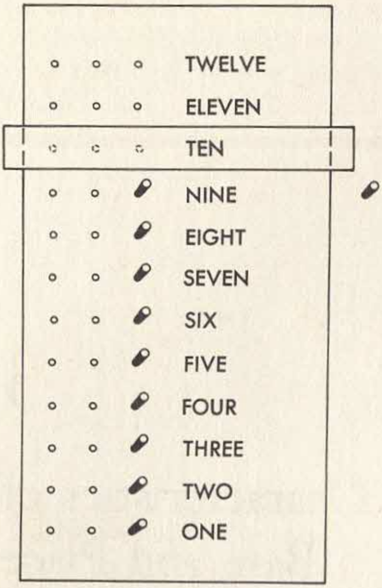


FIGURE 3–1

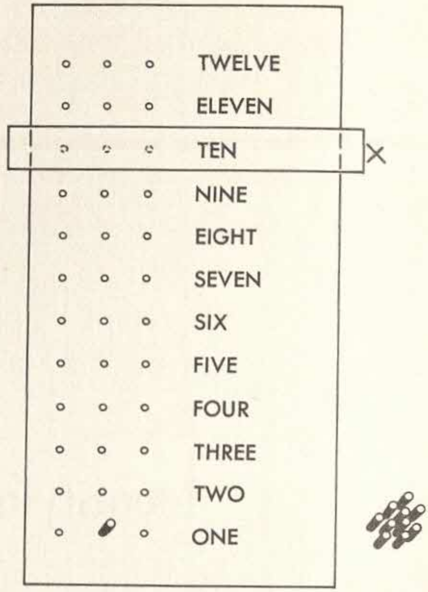


FIGURE 3–2

into a bundle until there were ten of them). Second, if we agree to put bundles aside *only* if they have ten pegs, we can see that it is a bundle of ten pegs by noting where we place it. If we can tell this is a bundle of ten by its position, let us use this property of position to keep the problem of counting from becoming unwieldy. We will put only *one* peg in the second row to indicate that we have *one* bundle of ten pegs already counted (Figure 3–2).

We have already made use of the two concepts around which this chapter is written—*base* and *place*. The base we have used is ten. This was the number of units we counted before a change was needed in our technique of denoting the number counted. After we had counted out the first ten pegs, we were forced to make a change because no tenth hole was available. The base in any system of numeration is the number of units which must be reached before a change is made in the pattern used to denote the numeral. The place to which we have referred is an actual physical location, relative to some starting position. The starting position in the above description is the right-most set of holes: those representing units.

Now if we go back to counting, when we get to ten pegs again, we will pick up all of the units pegs, bundle them together, and put a second bundle of ten off to the side and put a second peg in the second

row. We continue this process until we reach a peg which represents the tenth bundle of ten (Figure 3–3). There is no hole for this tenth peg, so we bundle up all ten of these pegs and put one peg in the next row (Figure 3–4) to show that we have reached a point in our counting at which we have ten bundles of ten pegs each. That is, a peg in this row represents one hundred pegs. This one hundred is ten tens, a fact which we will use shortly.

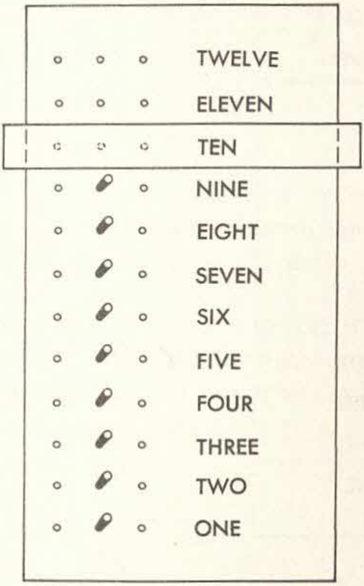


FIGURE 3–3

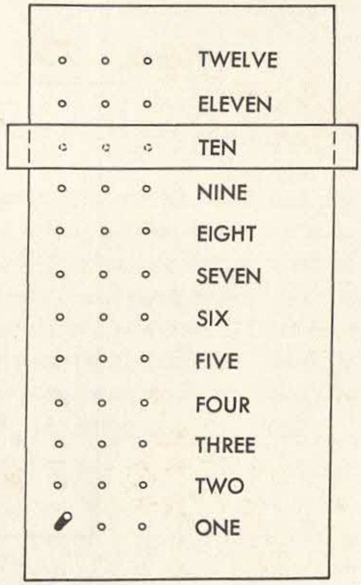


FIGURE 3–4

We could continue this process, but the pattern seems to be well established now. If we had pegs arranged as in Figure 3–5, we would be representing the number composed of three hundreds and two tens and six—that is, three hundred twenty-six. The pattern could be stated as $3 \times 100 + 2 \times 10 + 6$, or as $6 + 2 \times 10 + 3 \times 100$. Before introducing any abbreviations into our writing of numerals, we must agree on the meaning of our abbreviations. There is certainly nothing sacred about the abbreviated way in which we write numerals, but there is a conventional way to write them. This conventional pattern makes use of place value just as place was used on the peg board. The individual characters we use in writing a numeral are called digits; these include 0, 1, 2 . . . up through a digit representing a number which is one less than the base. For natural numbers, the digit in the right-most position

of the numeral indicates units and is called the ones or units digit. Moving toward the left, the second digit of a numeral in a base ten system indicates how many tens are contained in the number represented by the numeral. This is called the tens digit. Similarly, the next digit to the left is the hundreds digit, the next is the thousands digit, etc. The system of numeration with base ten is called the decimal system.

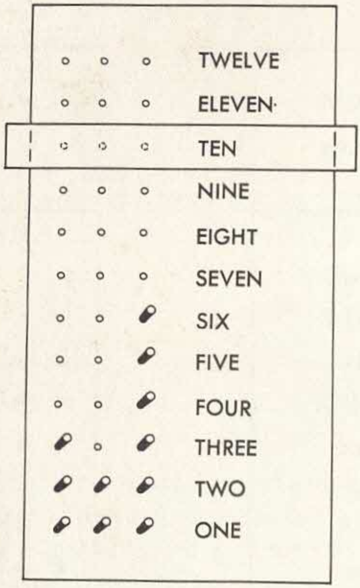


FIGURE 3–5

Returning to the previous example, let us write the number represented by $3 \times 100 + 2 \times 10 + 6$ as 326 to be in accord with the conventional pattern. We will generally arrange our numerals in this order, even when they are written out in a form like $3 \times 100 + 2 \times 10 + 6$ to display their meaning. Even though $3 \times 100 + 2 \times 10 + 6$ is equal to $6 + 2 \times 10 + 3 \times 100$, we would *not* want to write this number as 623 because of the conventions we have agreed to use when writing numerals.

It is of interest to note that the conventional order of writing a numeral (in the decimal system) may not be the most convenient order. In learning to name the number represented by the numeral 3278, we begin at the right-most digit and count to the left (silently) by units, tens, hundreds, thousands, etc., until we come to the leftmost digit before we are able to start reading aloud "three thousand, two hundred,

seventy-eight." It might actually be simpler to read eight, and seventy, and two hundred and three thousand.

Thousands	Hundreds	Tens	Ones
3	2	7	8

If it were well understood between writer and reader, could we substitute some other base for the base ten?

Let us try using five as the base. On the peg board we move our peg crosspiece to five (Figure 3–6) instead of ten as we had it for our base ten system. How many digits do we need to describe the number of pegs we have used? 0, 1, 2, 3, 4. We *choose* to call these digits by their familiar names and to denote them by their familiar symbols to reduce confusion. There is no necessity to do this; other symbols could be used, but since our purpose is to look at structure, nothing would be gained, and much confusion could result, by using new symbols.

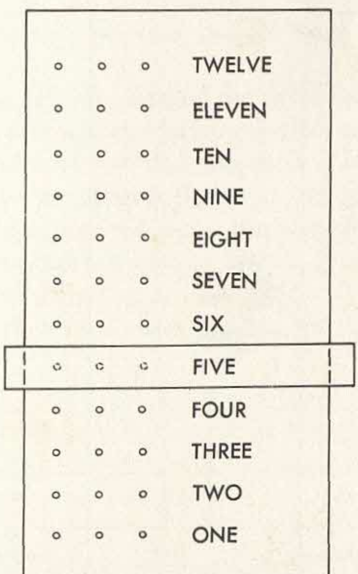


FIGURE 3–6

In this *quinary* system, the first four counting numbers are denoted by the numerals 1, 2, 3, and 4 (just as the first nine in the decimal system are denoted by 1, 2, 3, 4, 5, 6, 7, 8, 9). Since there is no hole in the first row for a fifth peg (Figure 3–7), we gather the five pegs and put a peg in the second row to indicate this one bundle of five. Logically, then we would denote the number ordinarily named five by 10. However, do not call this number by the name ten even though it looks like the numeral 10 of the decimal system. We are not talking about the number ten. Hereafter, we will call a *number* by its common name. We could write this as $(10)_{\text{five}}$. The word five is used here as a subscript to denote the base. This becomes tedious, so for the moment let us recognize that we are working in the quinary system and denote the number five by 10; 10 will mean one five and no units. Similarly, six would be denoted by 11, since this means one five and one unit.

Table I presents some correspondences. We now ask the questions, "What do we do now? How do we go on?" In the decimal system, when we had used up all digits in the first two places, what did we do? We went on to three digit numerals, with the left digit being the multiple of 100 (or 10^2) which was to be included in the number represented by the numeral. We do a similar thing here. On a peg board, this is very easy to illustrate (Figure 3–8). When we have accumulated five fives, we denote this by the digit 1 in the third position in our numeral.

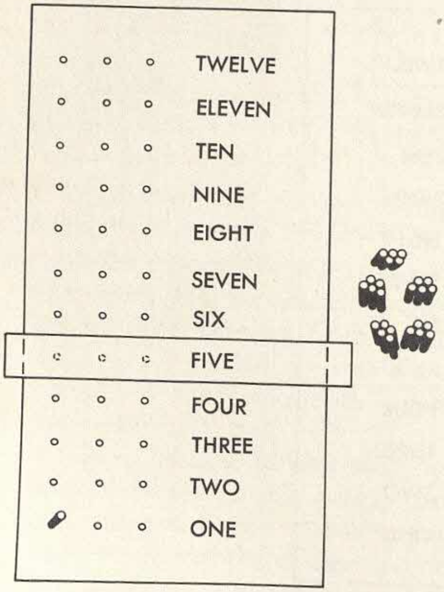


FIGURE 3–7

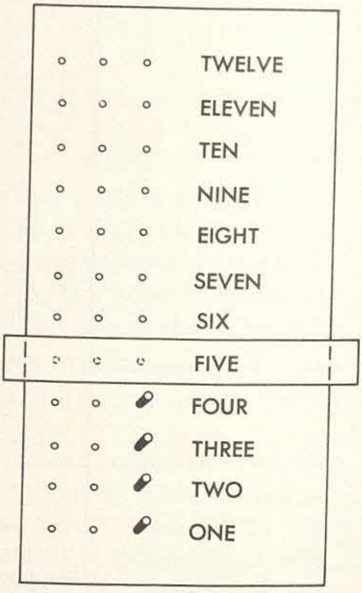


FIGURE 3–8

TABLE I

Base Ten Name	Quinary Numeral	Meaning of Numeral and Quinary Name
one	1	one
two	2	two
three	3	three
four	4	four
five	10	one five
six	11	one five and one
seven	12	one five and two
eight	13	one five and three
nine	14	one five and four
ten	20	two fives
eleven	21	two fives and one
twenty	40	four fives
twenty-four	44	four fives and four

That is, twenty-five is denoted by 100—meaning one group of five fives, no fives, and no units. Then twenty-six would be written 101, meaning one twenty-five, no fives, and one unit. Other illustrations are shown in Table II.

If the reasons are actually worked out before the quinary numeral is written, it is easy to see why the numeral takes the form it does. In fact, the *form* or structure is the same as it is in the decimal system. Only the base has been changed; the digits still have place value.

Many classroom teachers utilize the place value chart in helping boys and girls develop concepts of base and place. The place value chart can be used as readily in base five as in base ten. The typical place value chart would need revision so that pockets are named ones (or units),

Twenty-fives	Fives	Units

Base Five

Hundreds	Tens	Units

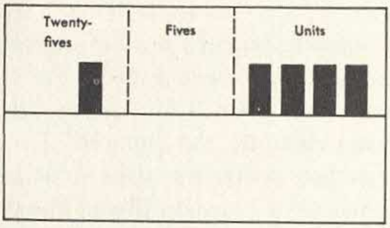
Base Ten

PLACE VALUE CHARTS

TABLE II

Base Ten Name	Quinary Numeral	Meaning of Numeral and Quinary Name
twenty-nine	104	one twenty-five and four
thirty	110	one twenty-five and one five
thirty-four	114	one twenty-five, one five, and four
thirty-eight	123	one twenty-five, two fives, and three
forty	130	one twenty-five and three fives
eighty-two	312	three twenty-fives, one five, and two
one hundred seventy-two	1142	one one-hundred twenty-five, one twenty-five, four fives, and two

fives, twenty-fives, and one hundred twenty-fives. When five markers fill the ones pocket, they are bundled together to become one in the fives pocket. In the illustration below, 104 appears in the place value chart as one twenty-five, no fives, and four ones.



Base Five

To see the structure more clearly and to free ourselves from dependence on the peg board or place value chart, the abacus or other devices, let us review briefly a few facts from algebra. The pertinent facts we need here are few and easy to grasp. The first of them is this: when, in algebra, we say that we are going to let a number be represented by the letter n (or a, or b, or any other letter), we are not implying that the numeral n has any particular structure. In particular, the numeral n is not to be interpreted as if we were applying the base and place characteristics of numerals with which we ordinarily work. The second

fact that we need is one from the topic of exponents: when we write n^3 we mean $n \times n \times n$, and we read this as n cubed or n to the third power. Similarly, n^2 means $n \times n$, and in the same pattern n^1 means n itself. To make the pattern complete, we let $n^0 = 1$. If this last definition seems strange, remember that it is used to complete the pattern developed for the meanings of other exponents.

$$104 = 1 \times f^2 + 0 \times f^1 + 4 \times f^0$$

$$110 = 1 \times f^2 + 1 \times f^1 + 0 \times f^0$$

$$114 = 1 \times f^2 + 1 \times f^1 + 4 \times f^0$$

$$123 = 1 \times f^2 + 2 \times f^1 + 3 \times f^0$$

$$130 = 1 \times f^2 + 3 \times f^1 + 0 \times f^0$$

$$312 = 3 \times f^2 + 1 \times f^1 + 2 \times f^0$$

$$1142 = 1 \times f^3 + 1 \times f^2 + 4 \times f^1 + 2 \times f^0$$

If we now let the letter f be the symbol which stands for the base five in the algebraic representation of a number (we do not use the symbol 5 for fear that some confusion between the algebraic structure and the existence of a single digit representing the number five in base five may occur), we can use the ideas of exponents to enable us to see the structure of the base five numerals which were used in Table II.

Remembering that we are using the ordinary (base ten) names for numbers, we can now show, by way of illustration, how the number one hunderd seventy-two was broken down in order that the numeral representing it be written in its quinary form. The arithmetic in the left column of the accompanying display is done in the familiar base ten numeral system, while the base five is still represented by the letter f. First, find the highest power of five which is contained in one hundred seventy-two. Certainly five is, and also five squared (or twenty-five) is contained. We find also that five cubed (one hunderd twenty-five) is contained, but that five to the fourth power (six hundred twenty-five) is not contained. Five cubed is contained only one time since the difference between one hundred seventy-two and one hundred twenty-five is smaller than one hundred twenty-five. Hence, five cubed

$$\begin{array}{r}
172 \\
-\ 125 \text{ or } 1 \times f^3 \\
\hline
47 \\
-\ 25 \text{ or } 1 \times f^2 \\
\hline
22 \\
-\ 20 \text{ or } 4 \times f^1 \\
\hline
2 \text{ or } 2 \times f^0
\end{array}$$

occurs *one* time. The difference between one hundred seventy-two and one hundred twenty-five is forty-seven.

Five squared (twenty-five) is contained only *one* time in forty-seven. The difference between forty-seven and twenty-five is twenty-two. Five is contained *four* times in twenty-two, and the remainder is *two* units. Hence, the numeral representing one hundred seventy-two in the quinary system is 1142. We may find that we are forced to do our calculations in the decimal system to discover the above facts, but this computational difficulty is caused only by our unfamiliarity with the quinary system and is not a problem inherent in the system.

ARITHMETIC IN OTHER BASES

Now let us work out an addition table. There are one hundred basic addition facts in decimal notation since the addition facts are related to the combination of any one of the ten digits with any other one.

ADDITION TABLE

(Base Five)

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

FIGURE 3–9

Many authorities include only 81 facts, discounting those facts which include zero plus any number or any number plus zero. The number of basic addition facts in the quinary system is five times five or twenty-five. These may be expressed in an addition table as illustrated in Figure 3–9. The addition fact is found at the intersection of the row and column indicated by the digits along the top and left edges of the table.

Suppose now we are called on to work an addition problem in the quinary system. We can refer to our addition table for the pertinent addition facts, and we will need to remember that while "exchanging" in this system means "exchanging" multiples or powers of five, the mechanics are identical with those in the decimal system. Let us add 24 and 33. The sum of 3 and 4 (the units digits) is 12, so we write down the 2 and exchange or "carry," as we frequently call it, the one (one five, that is). In the fives column we now must add 1, 2, and 3. The sum of 1 and 2 is 3, and the sum of 3 and 3 is 11, so we write down the 11 to obtain the sum 112. We might check as follows: 2 fives and 4 symbolize the number whose common name is fourteen (two fives and four units), and 3 fives and 3 symbolize eighteen. The sum of fourteen and eighteen is thirty-two which is symbolized by 112 (one twenty-five, one five, and two units).

$$\begin{array}{r} 1 \\ 24 \\ 33 \\ \hline 2 \end{array} \qquad \begin{array}{r} 24 \\ 33 \\ \hline 112 \end{array}$$

Let us try the problem below in which there are three addends. First we add 2, 1, and 4. Using the addition table, we find that the total is 12. We write down the 2 and carry the 1. Now we add 1, 3, 4, and 3. Using the addition table, we find that the total of 1, 3, and 4 is 13, but we must do a secondary problem of 13 + 3 to get this partial sum. This sum 13 + 3 is done as before to give 21. We now copy down the 1 and carry the 2 (in this case meaning 2 twenty-fives). The sum in the third column is the sum of 2, 1, and 2. Using the addition table, we find that this is 10 which we write down to complete the problem.

$$\begin{array}{r} \\ 32 \\ 141 \\ 234 \\ \hline \\ \end{array} \qquad \begin{array}{r} 1 \\ 32 \\ 141 \\ 234 \\ \hline 2 \end{array} \qquad \begin{array}{r} 21 \\ 32 \\ 141 \\ 234 \\ \hline 12 \end{array}$$

$$\begin{array}{r} 32 \\ 141 \\ 234 \\ \hline 1012 \end{array}$$

Next, let us consider the problem of subtraction. Since subtraction is the inverse of addition, we will again use the table of addition facts to perform subtractions. Subtract 13 from 34. No difficulties arise, and we see that the difference is 21 which is 2 fives and 1 unit. Now let us subtract 24 from 41. As in ordinary arithmetic, we must exchange or borrow. The one we borrow is really one five.

$$\begin{array}{r} 34 \\ -\ 13 \\ \hline 21 \end{array}$$

41 (4 fives and 1 unit)	4 fives + 1 = 3 fives + 1 five + 1 = 3 fives + 11
−24 (2 fives and 4 units)	2 fives + 4 = 2 fives + 4 = 2 fives + 4

In order to subtract 4 from 11, we need to look at the addition table to find that $4 + 2 = 11$. Hence, $11 - 4 = 2$, and $41 - 24$ is 12.

$$\begin{array}{r} 41 \\ -\ 24 \\ \hline \end{array} \qquad \begin{array}{r} 41 \\ -\ 24 \\ \hline 12 \end{array}$$

Exchanging may, of course, be done in any "place" in the numeral where it may be necessary.

We have seen how addition and subtraction are performed in this numeral system. Only limitations of space prevent us from directing our attention to problems of multiplication and division which can be done equally as well in the quinary system as in the decimal.

Numeral systems with several different bases are actually in use today. In the measurement of time and of angle, some use is made of base sixty. (A base sixty system is called a sexagesimal system.) In these systems, though, the use of the concept of place value is negligible. In our giant computing machines, the binary (base two) and octal (base eight) numeral systems are widely used, and if proper electronic components were available, ternary (base three) would be the preferred system. Other systems which find some use are the duodecimal (base twelve) and the sexadecimal (base sixteen). Our peg board can be used to demonstrate any base from 2 to 12.

In conclusion we may say that in order to communicate quantitative relations by use of numerals, we need to have a common means of writing and speaking these numerals. This community of understanding embraces both the base being used and the order of assignment of place value.

EXERCISES

1. What do our number names "hundred" and "thousand" represent in relation to sets of ten objects?

2. Express with positional numerals in the given base
   (a) base eight: 3 sixty-fours 7 eights 4 ones
   (b) base seven: 2 forty-nines 3 ones
   (c) base five: 2 hundred twenty-fives 3 twenty-fives 4 fives

3. Write the meanings of the following numerals in a pattern comparable to the statements of parts (a)-(c) of Exercise 2.
   (a) base nine: 347
   (b) base four: 2121
   (c) base eight: 7352

4. What base ten numeral will represent each number symbolized in Exercises 2 and 3?

5. In a base five positional value numeral system, how many digits are there?

6. In this same system, how many two digit numerals are there with zero in them?

7. In base ten, what is the largest number that can be represented with the numerals 1, 3, 5, 5, 9, 0? What is the smallest number that can be denoted?

8. Can our modern addition algorithm be used to add
   (a) in numeral systems with bases other than ten?
   (b) with a nonpositional numeration system?
   (c) *only* Hindu-Arabic numerals?

9. Suppose you travel to a mythical country where they use a positional system of numeration with a base other than ten. You enter a store and select three oranges which are priced at 3¢ each. The clerk correctly charges you 21¢. What base does this country use in its system of numeration? Explain.

10. If we use the place value chart to help pupils understand the concept of "base three", we would have to change the name of the "hundreds" pocket to the "nines" pocket. What name should we give this pocket in each instance below?
    (a) base two
    (b) base seven
    (c) base twelve
    (d) base sixteen

11. Convert 57_{ten}, 391_{ten}, 126_{ten} to binary notation.

12. Change the following binary numerals to base ten numerals.
    (a) 110101
    (b) 110011
    (c) 11011101
    (d) 11111111
    (e) 1000001
    (f) 1001001

13. Change the following base ten numerals to binary notations.

(a) 9	(c) 31	(e) 100
(b) 21	(d) 65	(f) 128

14. Represent the number which has the base ten numeral 5327 in:

(a) base eight
(b) base seven
(c) base twelve

15. Represent the number which has the base eight numeral 5327 in:

(a) base ten
(b) base two
(c) base five

16. Draw representations of place value charts, label the pockets, and with marks show how markers would be used to show the number 536_{ten} in:

(a) base five
(b) base eight
(c) base twelve

Using the addition table on page 49 as an aid, find the answers to the following exercises in base five.

	(a)	(b)	(c)	(d)	(e)
17.	$\begin{array}{r} 2 \\ +\ 2 \\ \hline \end{array}$	$\begin{array}{r} 3 \\ +\ 4 \\ \hline \end{array}$	$\begin{array}{r} 12 \\ -\ 3 \\ \hline \end{array}$	$\begin{array}{r} 23 \\ +\ 11 \\ \hline \end{array}$	$\begin{array}{r} 43 \\ +\ 24 \\ \hline \end{array}$
18.	$\begin{array}{r} 34 \\ -13 \\ \hline \end{array}$	$\begin{array}{r} 41 \\ -\ 24 \\ \hline \end{array}$	$\begin{array}{r} 32 \\ +\ 13 \\ \hline \end{array}$	$\begin{array}{r} 102 \\ -\ 34 \\ \hline \end{array}$	$\begin{array}{r} 1341 \\ +\ 223 \\ \hline \end{array}$
19.	$\begin{array}{r} 24 \\ 31 \\ +10 \\ \hline \end{array}$	$\begin{array}{r} 201 \\ 4 \\ +\ 32 \\ \hline \end{array}$	$\begin{array}{r} 42 \\ 334 \\ +\ 12 \\ \hline \end{array}$	$\begin{array}{r} 21 \\ 212 \\ +2121 \\ \hline \end{array}$	$\begin{array}{r} 4004 \\ 314 \\ +\ 122 \\ \hline \end{array}$

20. Find the answers to the following base two exercises.

(a)	$\begin{array}{r} 11 \\ +\ 1 \\ \hline \end{array}$	$\begin{array}{r} 101 \\ +\ 11 \\ \hline \end{array}$	$\begin{array}{r} 1101 \\ +\ 11 \\ \hline \end{array}$	$\begin{array}{r} 10101 \\ +\ 11 \\ \hline \end{array}$	$\begin{array}{r} 1011 \\ +\ 111 \\ \hline \end{array}$
(b)	$\begin{array}{r} 111 \\ -\ 11 \\ \hline \end{array}$	$\begin{array}{r} 101 \\ -\ 10 \\ \hline \end{array}$	$\begin{array}{r} 10111 \\ -\ 1011 \\ \hline \end{array}$	$\begin{array}{r} 10101 \\ -\ 111 \\ \hline \end{array}$	$\begin{array}{r} 1001 \\ -\ 111 \\ \hline \end{array}$
(c)	$\begin{array}{r} 11 \\ \times\ 10 \\ \hline \end{array}$	$\begin{array}{r} 110 \\ \times\ 11 \\ \hline \end{array}$	$\begin{array}{r} 1001 \\ \times\ 101 \\ \hline \end{array}$	$\begin{array}{r} 111 \\ \times\ 111 \\ \hline \end{array}$	$\begin{array}{r} 10101 \\ \times\ 101 \\ \hline \end{array}$
(d)	$11\overline{)11}$	$10\overline{)1100}$	$110\overline{)111010}$	$111\overline{)10111}$	$101\overline{)11110}$

21. Construct an addition table for base eight. Use the facts from this table to work these exercises which are stated in base eight.

(a) $\begin{array}{r} 7 \\ +5 \\ \hline \end{array}$ (b) $\begin{array}{r} 6 \\ 3 \\ +2 \\ \hline \end{array}$ (c) $\begin{array}{r} 5 \\ 6 \\ 7 \\ +2 \\ \hline \end{array}$ (d) $\begin{array}{r} 17 \\ -\ 6 \\ \hline \end{array}$ (e) $\begin{array}{r} 12 \\ -\ 7 \\ \hline \end{array}$ (f) $\begin{array}{r} 34 \\ -\ 5 \\ \hline \end{array}$

22. In base ten, show why, when the last digit of the numeral is 0, 2, 4, 6, 8, the number is divisible by 2; when the last digit is 0 or 5 the number is divisible by 5.

23. (a) In base two notation, how are even and odd numbers distinguished?
(b) In base two notation, how do you know when a number is divisible by four?

24. *True or False*

(a) There are 110_{three} eggs in a dozen.
(b) Water boils at 3130_{four} degrees Fahrenheit.
(c) There are 263_{twelve} days in a year.
(d) A football team fields 1011_{two} players.
(e) There are more number facts in base two than in base ten.
(f) In base seven multiplication, a number that is exchanged has the same value as the one exchanged in base ten.
(g) In the United States, a man must be 110_{five} years old to become President.

25. Write 7^{10} using a positional numeral.

26. Write 5^8 using a positional numeral.

27. Express the following in scientific notation:

(a) 1492
(b) 6,000,000
(c) 8 billion, 9 hundred million
(d) 750 trillion

28. Work the following problems using base five, base eight, and base two notations. The problems are stated using decimal notation for the numbers given in Exercise 11.

(a) $\begin{array}{r} 391 \\ +\ 57 \\ \hline \end{array}$ (b) $\begin{array}{r} 391 \\ -126 \\ \hline \end{array}$ (c) $\begin{array}{r} 126 \\ \times\ 57 \\ \hline \end{array}$

29. Convert the following base ten numerals to base five notation; to base three notation; to base twelve notation.

Base Ten	Base Five	Base Three	Base Twelve
1			
2			
3			
4			
5			
6			
10			
11			
12			
16			
21			
25			
26			

30. What digits are needed to work in base four?

31. What are the probable reasons for our use of a base ten numeration system?

32. Construct an addition table for a base seven numeration system.

33. Construct a multiplication table for base four.

Extended Activities

1. Give as many reasons as you can for the United States converting to a duodecimal base numeration system.

2. (a) What is a googol?
(b) How can it be symbolized with exponent notation?
(c) What is a googolplex?
(d) Can it be symbolized with exponents? How? (See pp. 2007-2009 of *The World of Mathematics* by James R. Newman.)

3. Construct a place value pocket chart for base seven. Write a complete report on how it can be used to represent numbers, the relation of the chart representation to the base seven numerals, and how it can be used to illustrate addition using base seven numerals. Include a sequence of examples which could be used in teaching the meaning of base and place by use of base seven.

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Chapter 4

Concepts to be developed in this chapter are:

1. *There are some basic properties of arithmetic which govern and simplify our computation.*
2. *These properties include:*
   a. *In addition and multiplication, interchange of order is permissible (commutativity).*
   b. *In addition and multiplication, grouping is permissible (associativity).*
   c. *Multiplication distributes with respect to addition.*
3. *These properties are not universally applicable.*

4

Applying the Properties of Arithmetic

We use some fundamental properties of arithmetic each time we work with a situation which involves number. An understanding of these properties provides flexibility as we work in numerical situations. There are many properties of arithmetic, but we will consider only five in this chapter.

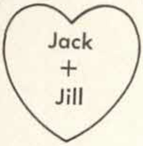
THE COMMUTATIVE PROPERTIES

The first property we discuss is the *Commutative Property of Addition.* This property is that $a + b = b + a$. This does not say that the arrangement of these numerals $a + b$ and $b + a$ is the same, nor that the arrangements of the physical situations represented by the numbers referred to by $a + b$ and $b + a$ are the same. It does mean, however, that the number which is associated with the combined value of a and b

is the same as the number associated with the combined value of b and a—that is, the total is the same.

The question then is: "How do we *know* that $3 + 4 = 4 + 3$?" We can't even begin to answer the question until we are sure we know what each of the components of the question means.

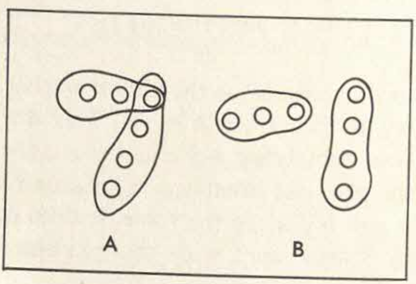
Consider the + sign. What does it mean? Its meaning may differ from one context to another, as in $3 + 4$ or



Of course, we are primarily interested in its meaning when it is used to connect or join together two numerals. Even here, though, we must be careful to distinguish between the purpose of the + sign in connecting two numerals and its purpose in joining (or conjoining) the members of the sets of objects which the numerals represent.

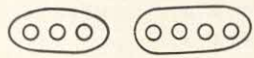
This last statement brings us to a very important matter in modern mathematics: the notion of set. One of the modern approaches to our number system and its properties uses set theory. What is a set? It is a *collection* of objects, ideas, etc. Associated with each set is a "label" or "tag" called a number, and this number is associated with every other set, the members of which can be put into one-to-one correspondence with the members of the given set. We use a numeral to describe in ordinary language the number (how many) of members possessed by each of these sets. In this fashion, number is associated with set. When two sets are brought into conjunction to form a new set, this new set has a number (the number of its members). This idea of conjoining sets can be used to describe addition or the meaning of the + sign.

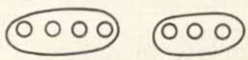
With this brief explanation, let us return to the question: "Is $3 + 4 = 4 + 3$?" Let us consider the two sets described below in A by three encircled pegs and four encircled pegs. Here are two sets. One has three members and the other has four members. The number associated



with the set made up of all members is six. In the two sets in B, though, we find that the conjoining of the sets (technically called the union of the sets) has the number seven describing the elements of the union. We can see, then, that unless we are a little more precise about the meaning of +, we cannot always say that $3 + 4 = 4 + 3$, or even that $3 + 4 = 3 + 4$. We should go back, then, and say that the sum of two natural numbers (counting numbers) is the number of elements in a set which is the union of two *disjoint* sets (no common members) having as their number of members the two given numbers.

This will, of course, rule out the case of  and will leave only

that $3 + 4$ represents 

and $4 + 3$ represents 

Both of these sets have the same number (of members), and consequently we have $3 + 4 = 4 + 3$.

We have discussed this commutative property of addition for natural numbers a and b. It may be extended to any of the numbers with which we are familiar. It is a property of ordinary arithmetic and one which we are accustomed to using daily, but it is far from being intuitively evident.

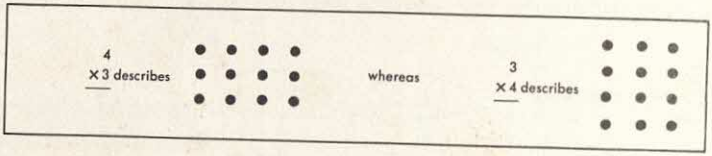
Elementary school children use the commutative property of addition when they reverse the order of the addends to check their work. Likewise, in studying addition facts, children learn that the reverse of an already known fact is the same as the original fact.

Without dwelling here on the meaning of multiplication, we recognize that we use a corresponding property of multiplication. It is, if a and b are numbers, then $ab = ba$. (We should remember that a times

	Check				
21	32				
32	21	4	3	7	2
53	53	+3	+4	+2	+7
	Check				
4	7	6	8	1	9
3	3	+8	+6	+9	+1
7	4	14	14	10	10
14	14				

b may be written as $a \times b$, $a \cdot b$, or merely ab. We do not always use the first form since the $\times$ may be confused with the letter x.) This property of arithmetic is called the *Commutative Property of Multiplication.*

When multiplication facts are taught in pairs, the commutative property is being applied. Textbooks frequently ask pupils to work each example, such as the two below, and write the fact that goes with it. Children should be made aware of the idea that these answers are the same even though the situations described may be different.



The commutative property of multiplication is also used for convenience. Given the problem (illustrated below) of finding how much party money the 32-member class will have when each child brings his nickel, the number of children is the multiplier and the 5¢ is the multiplicand. The question asked is, "How much is 32 nickels?" When we work this problem, we reverse it for convenience to five 32's. The use of the property of commutativity is even more convenient as the numbers involved become larger. Each of 375 children in the cafeteria spends 25¢. What is the total amount paid into the cafeteria?

$$\begin{array}{r} 5¢ \\ \times 32 \\ \hline 10 \\ 15 \\ \hline 160¢ \end{array} \qquad \begin{array}{r} 32 \\ \times\ 5¢ \\ \hline 160¢ \end{array} \qquad \begin{array}{r} 25 \\ \times 375 \\ \hline \end{array} \qquad \begin{array}{r} 375 \\ \times\ 25 \\ \hline \end{array}$$

As a memory device, we might remember the word "commutative" by associating it with this idea: a commuter is a person who goes *back and forth* by some means of conveyance; that is, he interchanges his place of abode for his place of work and vice versa. In a like manner, the a and b change places in the statement of the commutative property. In other sections, we will be dealing with the fundamental processes of addition, multiplication, subtraction, and division. It should be mentioned here that addition and multiplication are the primary processes and that subtraction and division are the secondary processes—

secondary in the sense that they are defined in terms of the primary processes. Since they are merely defined in terms of addition and multiplication and are not identical with these operations, there would be no reason to expect the same properties to apply to them. In fact, we see that the commutative properties of addition and multiplication are not universally true merely by turning to subtraction and division in ordinary arithmetic. It is quite obvious that the principle of commutativity does not apply to subtraction or division—that is, $7 - 4 \neq 4 - 7$, and $8 \div 2 \neq 2 \div 8$. (The symbol $\neq$ signifies "is not equal to.")

THE ASSOCIATIVE PROPERTIES

We turn to another property of arithmetic, the *Associative Property of Addition.* This property may be stated as follows: if a, b, and c represent numbers, then $a + b + c = a + (b + c) = (a + b) + c$. That is: numbers may be associated in addition (without change of order) in any combination without changing the value of the total. An example of this principle can be seen in $7 + 3 + 2 = 7 + (3 + 2) = (7 + 3) + 2$.

Many people add columns of figures using the associative property. In the following example of column addition, a person might collect tens mentally as he adds.

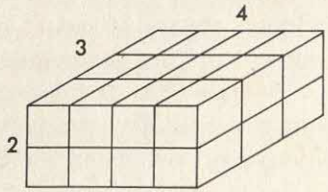
$$\begin{array}{r}6\\4\\3\\5\\2\\7\\3\\8\\ \hline 38\end{array}\quad\begin{array}{l}\\ \\ \\ \\ \\ \\ \\ \left\{\begin{array}{l}\text{3 tens and}\\ \text{8 ones}\end{array}\right.\end{array}$$

In mental computation the associative and commutative principles are of much help. For example: 46 plus 35. Using the commutative property, we mentally interchange the 6 and the 30; next, using the associative property, we add the 40 and the 30 (yielding 70), then to 70 add the $6 + 5$ (which is 11) and get 81.

We see that when we combine this property with the commutative property of addition, terms may be associated and combined in any order without changing the value of the total.

$$
\begin{aligned}
46 + 35 &= 40 + 6 + 30 + 5 \\
&= 40 + 30 + 6 + 5 \\
&= 70 + 11 \\
&= 81
\end{aligned}
$$

The *Associative Property of Multiplication* for numbers a, b, and c is: $abc = a(bc) = (ab)c$. Again, as in the case of addition, this property can be combined with the commutative property of multiplication to assure the same product when the factors are associated in any order whatever. This property is used sometimes in finding the volume of a rectangular solid, where we know that the product of the dimensions is the same, independent of which pair of dimensions is multiplied first.



$$2 \times 3 \times 4 = 2 \times (3 \times 4) = (2 \times 3) \times 4$$

An interesting example showing that the associative properties are not universally true is given by Mueller in *Arithmetic: Its Structure and Concepts.*[1] He says that at a ball park in the summer it is not unusual to see someone eating a hot dog with mustard and drinking soda pop. Consider whether, from the standpoint of taste, the association property holds. Would [(hot dog + mustard) + soda pop] = [hot dog + (mustard + soda pop)]?[1]

Like the commutative property, the associative properties are not universally true. Again, for example: in subtraction and division $3 - (5 - 2) \neq (3 - 5) - 2$ and $18 \div (36 \div 4) \neq (18 \div 36) \div 4$.

THE DISTRIBUTIVE PROPERTY

Now we come to the property of arithmetic which joins the two processes of addition and multiplication. It is called the *Distributive*

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[1] Francis J. Mueller, *Arithmetic: Its Structure and Concepts* (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1956), p. 24.

Property (or, more properly, the *Distributive Property of Multiplication with Respect to Addition*). This property may be stated as $a(b+c) = ab + ac$.

$$a(b+c) = ab + ac$$

$$3(40+2) = (3 \times 40) + (3 \times 2)$$
$$= 120 + 6$$
$$= 126$$

$$\begin{array}{r} 42 \\ \times\ 3 \\ \hline \end{array}$$

This can be shown simply in a multiplication example with a two place multiplicand and a one place multiplier. The multiplier "a" is distributed among the added terms. (The reason for the longer title given a moment ago is that we could state a distributive property of addition with respect to multiplication. It would be $x + yz = (x+y)(x+z)$, but in ordinary arithmetic it is only true when $x = 0$ or when $x + y + z = 1$. Since it is not universally true, even for ordinary numbers, it is not stated as a property for ordinary arithmetic. However, some branches of advanced algebra do use such a property.)

These five properties may be organized as shown in the table below.

	ADDITION	MULTIPLICATION
Commutative Property	$a + b = b + a$	$ab = ba$
Associative Property	$a + b + c = (a + b) + c$ $= a + (b + c)$	$abc = (ab)c = a(bc)$
Distributive Property	$a(b + c) = ab + ac$	

EXERCISES

1. Add in base ten $7 + 3 + 6 + 2$. Check by adding $2 + 6 + 3 + 7$. Which of the five properties of arithmetic have you used? Discuss.
2. Does $.04 + .70 = .70 + .04$? What property is illustrated by this equation?

3. Does the associative property operate with decimal fractions? Illustrate this property using decimal fractions.

4. Is base five multiplication commutative? Explain.

5. Give illustrations to show that subtraction is not commutative; that division is not commutative.

6. Give illustrations to show that subtraction is not associative; that division is not associative.

7. Illustrate that multiplication is distributive with respect to subtraction; that division is distributive with respect to addition; that division is distributive with respect to subtraction.

8. (a) Multiply $6(7+4)$ using the distributive property; then collect.
   (b) Add $7+4$; then multiply by 6.
   (c) Is the answer the same?

9. If a student know the products 5×4 and 5×3, show how he can find the product 5×7.

10. When dividing 138 by 6, the problem may be approached as: $138 \div 6 = (120 \div 6) + (18 \div 6)$. Which of the five properties of arithmetic discussed in Chapter 4 is being used in this simplification?

11. Show how the distributive property may be used in teaching a child how to:
    (a) divide 369 by 3.
    (b) divide 4832 by 16.

12. Show how the distributive property helps in solving the exercise $27 \times 39 = N$.

13. Use the distributive property twice to multiply 23×47. Then collect.

14. Show the relation of the steps in problem 13 to the partial products in:

(a)
$$\begin{array}{r} 23 \\ \times\ 47 \\ \hline 21 \\ 140 \\ 120 \\ 800 \\ \hline 1081 \end{array}$$

(b)
$$\begin{array}{r} 23 \\ \times\ 47 \\ \hline 161 \\ 92 \\ \hline 1081 \end{array}$$

15. Multiply 27 by 43. Discuss all uses of the five properties and the base and place characteristics of the numerals used.

16. Do the five properties depend on the use of positional-type numerals? Discuss.

17. Using a pile of stones or other objects, demonstrate the communicative property of arithmetic for addition. The associative property. The distributive property.

18. Show, using these same markers, that the communicative and associative properties do not operate in doing subtraction and division.

Select the appropriate response to each of the following statements and explain the reason for your choice.

19. Which of the following properties are not appropriate to a description of the standard algorithm in obtaining the product in the following problem?

$$\begin{array}{r} 32 \\ \times\ 5 \\ \hline \end{array}$$

(a) commutative property of multiplication
(b) distributive property of multiplication with respect to addition
(c) associative property of multiplication

20. Which of the following problems will require the use of the associative property of multiplication in its solution?

(a) $6 \times 8 \times 7$
(b) $(6 + 7) \times (4 + 9)$
(c) $3(11 + 9) + 31$

21. The fact that the relative position of the numerals in the column addition shown below does not affect the sum is an example of which of the following properties of arithmetic?

$$\begin{array}{r} 32 \\ 59 \\ \underline{27} \end{array} \qquad \begin{array}{r} 32 \\ 27 \\ \underline{59} \end{array}$$

(a) commutative (b) associative (c) distributive

22. Which of the following problems will require the use of the distributive property of multiplication in its solution?

(a) 6×4
(b) $2 \times 3 \times \frac{1}{4} \times 6$
(c) $2\,(2 + 3)$
(d) 15×61

23. Which of these properties is not appropriate to addition?

(a) distributive property
(b) commutative property
(c) associative property

24. Which of the following statements is true of the algorithm we use for addition?

(a) Indiscriminate interchange of digits is allowed.
(b) The relative position of the digits in each vertical column has no effect upon the result.
(c) The units digits are always added first.
(d) The associative property is always used.

25. In which of the following steps does a student employ the associative property of addition to work a problem by the standard addition algorithm?

(a) carrying in the addition of two-digit addends
(b) arranging digits in vertical column according to place value
(c) adding in column addition
(d) changing the sum to algebraic form

26. A teacher using eight beads on an abacus first shows them in the manner illustrated in line A.

A. ...0000..........0000...

Then she arranges them as shown in line B.

B. ...00..........00..........00..........00...

Which of the following properties of arithmetic may she be demonstrating?

(a) commutative property of multiplication
(b) associative property of multiplication
(c) distributive property of multiplication with respect to addition

27. A student in an algebra class is given the following problem and told to remove the parentheses and combine like terms. He solves the problem as indicated:

$$\begin{aligned}&x(a + b) + ax\\&xa + xb + ax\\&ax + bx + ax\\&ax + ax + bx\\&2ax + bx\end{aligned}$$

Which of the following list of the basic properties of arithmetic most nearly describes all of the properties he employed in solving the problem?

(a) distributive
(b) associative and distributive
(c) distributive and commutative
(d) commutative, associative, and distributive

Extended Activities

1. Read *Basic Concepts of Elementary Mathematics* by William L. Schaaf, pp. 97-118, and report on how the five properties of arithmetic treated in Chapter 4 of *Understanding the Number System* fit into Schaaf's axioms for the natural numbers.

2.

PATTERNS IN THE MULTIPLICATION MATRIX

	0	1	2	3	4	5	6	7	8	9
0										
1		1								
2			4							
3				9						
4										
5										
6										
7										
8										
9										

Complete the multiplication matrix above. What patterns and generalizations can you find in the matrix? As a guide to your discoveries, complete the matrix in the following steps.

(1) Multiply each number by itself and write the products in the matrix. The first four products are illustrated above. Do you find any relationship among these products?

(2) Fill in all the products where 0 is one of the factors. What generalization can you make about 0 as a multiplier or a multiplicand?

(3) Fill in all the products where 1 is one of the factors. What generalization can you make about these facts?

(4) The multiplier is shown in the vertical column. The multiplicand is in the horizontal row of the matrix. Fill in the products for the following facts:

a. $3 \times 4 = K$ c. $8 \times 6 = L$ e. $2 \times 5 = M$ g. $4 \times 8 = N$
b. $4 \times 3 = K$ d. $6 \times 8 = L$ f. $5 \times 2 = M$ h. $8 \times 4 = N$

What property is displayed by these exercises?

(5) Fill in all the products where 5 is one of the factors. What number pattern do you find?

(6) Fill in all the products where 9 is one of the factors. What patterns do you find within the nines table?

(7) Fill in the remainder of the products in the matrix. What other number patterns do you find?

(8) Circle all products that are odd numbers. What are the factors of these products? What are the factors of the even-numbered products? What generalization can be drawn from this? How many of the 100 multiplication facts have odd-numbered products?

(9) Take a parting look at the products for these exercises:

$$6 \times 6 = K$$
$$5 \times 7 = L$$
$$4 \times 8 = M$$
$$3 \times 9 = N$$

Do you find a pattern?

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Chapter 5

Concepts to be developed in this chapter are:

1. *Elementary school children use many kinds of numbers.*
2. *The definition we use for the addition of one kind of number may be inadequate for another kind of number.*
3. *The addition algorithms we use have evolved from more complex arrangements of numerals.*

5

Understanding the Four Fundamental Processes

I. ADDITION

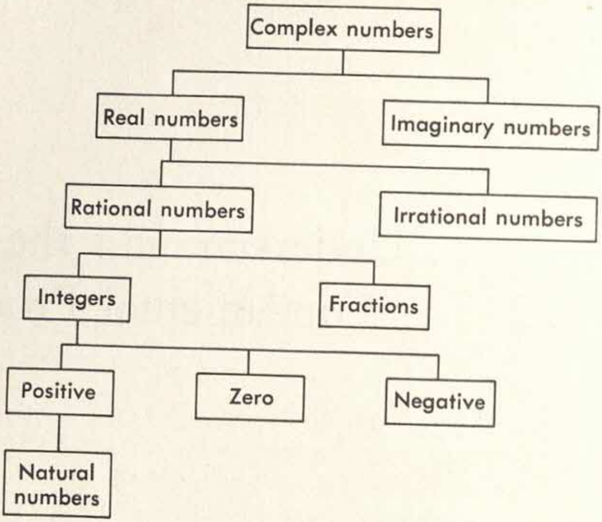
DEFINITIONS OF ADDITION

What is the nature of addition? How can we define it? What sort of operation is it? We are going to look at some answers to these questions; we will also look at some of the consequences, some of the outgrowths, and some of the obscure byways which have their beginnings in our conceptions of addition.

We could find several definitions of addition in the various textbooks being used in elementary, high school, and college classrooms today, and we would find, if we examined these definitions, that there are some elements of agreement and some elements of disagreement among them. Since each is a definition, we cannot say that one is right and one is wrong, but we might say that one is convenient and productive

of further consequences while another is not; or that one is definitive while another is not. Instead of investigating all possible definitions, we will take a careful look at the elements of one or two of them.

Before doing this, however, let us examine the kinds of numbers that various definitions must consider. We will recognize that the following table is concerned with numbers and that we use a variety of numerals to represent these numbers.[1]



The natural numbers are the counting numbers. These natural numbers are the ones with which children are first acquainted. The number zero is likewise one with which the child becomes familiar at an early age. During his first three years of school, the child works primarily with natural numbers (which are positive) and zero. At the fourth grade level, his concern with fractions increases, and by the time he reaches the sixth grade, some attention is given to the development of concepts of negative numbers. These first experiences with negative numbers are usually related to the reading of temperatures below zero. Children have some fraction concepts when they come to school, but intensive work with fractions does not begin until fifth grade. Fractions include the rational numbers—those numbers which *precisely represent quantitative situations.*

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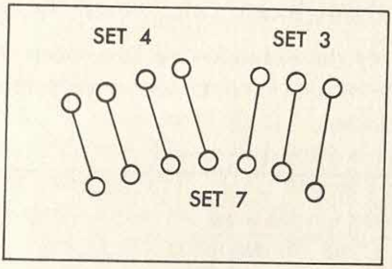
[1] William L. Schaaf, *Basic Concepts of Elementary Mathematics* (2nd ed.; New York: John Wiley & Sons, Inc., 1965), p. 227.

Swain defines the sum of two natural numbers as follows: "Given any two natural numbers, consider a pair of sets which represent them. The sum of the numbers is defined to be the number of the union of the sets."[2] This first seems to say that we need to find sets which are represented by the two given natural numbers, find another set which can be put into one-to-one correspondence with the members of the first two sets, and then find the natural number associated with this third set. What about this? Must we always conjoin the first two sets? Would we even think that sets must be brought into the definition at all?



Several textbooks give, with minor deviations, the following definition of addition of numbers: "Addition is the process of finding, without counting, a single number equal in value to the combined value of two or more others numbers." Let us look at this definition for a minute. First, we can see that it is not restricted to natural numbers, so in this sense it is more general than the first definition given; still, let us see if it may not raise more questions than it settles. What questions does it raise? Is it equally applicable to all types of numbers in contrast with the other definition which was restricted to natural numbers? Even if it is applicable to natural numbers, must we find the result, the sum, without counting? Having found the single number which is the sum, how do you know it is equal in value to the combined values of the other numbers? What is meant by "values"? If the numbers are not natural numbers, how does one determine what the "value" of each of the numbers is in order to find a combined "value"?

From these questions, we hope it is clear that it may be best first to define addition of natural numbers and that it is probably best to do this in terms of sets which represent these natural numbers.

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[2] Robert L. Swain, *Understanding Arithmetic* (New York: Holt, Rinehart & Winston, Inc., 1957), p. 47.

EXTENDING ADDITION TO OTHER NUMBERS

If we adopt this procedure and definition for the addition of natural numbers, we are faced with other questions. If we have defined addition only for natural numbers, then how do we add other types of numbers? The answer to this question is quite easy to give. We all know how to add common fractions, for example, but not everyone is aware that the reason we add them in a particular fashion is this definition: if a, b, c, and d are natural numbers, then the sum of $\frac{a}{b} + \frac{c}{d}$ is given by $\frac{ad + bc}{bd}$. We take this to be our definition of the sum of two common fractions because for the examples we have been able to add by some intuitive means, the results we obtained agree perfectly with the result given by the definition.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

$$\frac{1}{4} + \frac{2}{4} = \frac{4 + 8}{16} = \frac{12}{16} = \frac{3}{4}$$

or

$$\frac{1}{4} + \frac{2}{3} = \frac{3 + 8}{12} = \frac{11}{12}$$

To follow this system for each new type number introduced into our number system (negative integers, negative common fractions, decimal fractions, both positive and negative, and irrational numbers), we would need to define addition, in a manner relative to these numbers, by defining a procedure by which we would add these numbers. In fact, this is actually what we do as we develop the number system from year to year in our classrooms.

In the elementary grades we are primarily concerned with addition of natural numbers and some rational numbers (proper and improper common fractions and decimal fractions). Once in a while we are faced with a situation in which negative numbers appear, however, we will omit further consideration of negative numbers until the subject of subtraction is considered. It is so seldom that irrationals appear in the elementary grades we will hereafter omit them also.

ALGORITHMS FOR ADDITION

We have defined addition of natural numbers and of common fractions. (No real mention of decimal fractions has been made since every terminating or non-terminating repeating decimal may be put into the form of a common fraction for theoretical considerations. It is true, though, that we actually develop rules for adding decimal fractions without having to convert them to common fractions.) Our definitions, along with our assumptions of closure and completeness (see p. 154), allow us to develop work outlines (or algorithms as they are called) for accomplishing addition. The type of work outline–algorithm–we use for the addition of natural numbers is familiar to us all. We ordinarily write the numerals which represent the numbers to be added in a column (although this is not necessary) with all of the units digits in one vertical column, all of the tens digits in the next vertical column (working toward the left), etc. That is, we write the numerals in standard order, being sure that the units digits are in a column. We then begin by adding the units digits, the tens digits, etc. This is so familiar an algorithm that we are likely to forget all of the rules upon which the algorithm is based. Before looking at the algorithm itself, let us look at the rules (or axioms, or laws) on which its operation is based.

In order to look at these rules or laws let us take the problem: add 42 and 35. First, $42 = 40 + 2$ and $35 = 30 + 5$, so $42 + 35 = 40 + 2 + 30 + 5 = 40 + 30 + 2 + 5 = 70 + 7 = 77$. What rules have we used? We certainly used the following: (1) the principle of completeness (that the pair of numbers represented by 42 and 35 could be combined in this operation of addition), (2) the principle of closure (that the sum of the numbers represented by 42 and 35 actually existed within the system of numbers), (3) the rule for composition of numerals in the decimal system by which we know that 42 means $4 \times 10 + 2$ (which in itself assumes the rules of completeness, closure, multiplication of a digit by the base, and possibly others), (4) the commutative property for addition (that $2 + 30 = 30 + 2$ is a consequence of this law), (5) the associative property for addition (that $40 + 30 + 2 + 5 = (40 + 30) + (2 + 5) = 70 + 7$ is a consequence of our ability to associate the numbers represented by the numerals in any order). These last two properties were discussed at length in the preceding chapter.

After we have accepted the fact that there are many rules of arithmetic tied up in the addition algorithm even for natural numbers, let us look at the algorithm itself. If we want to add 42 and 35, we write the numerals down in the standard manner (indicated earlier) and then proceed to add units digits and then tens digits. In order to do this we

need to know some basic addition facts. This brings us face to face with another basic facet of addition—we must know these number facts about the addition of natural numbers before we can effectively use an algorithm. Since neither of the additions of elements in this problem gives a sum greater than ten, no special techniques are called for. We obtain 77 as shown in the illustration below.

$$\begin{array}{r} 42 \\ +\ 35 \\ \hline 77 \end{array}$$

If we try to add 47 and 35, though, we need an additional technique, that of carrying. We all recognize that the 1 which is carried really represents one ten which is then included in the sum of the tens digits to give a final sum of 82. This algorithm was devised in fairly modern times.

$$\begin{array}{r} 1 \\ 47 \\ 35 \\ \hline 82 \end{array}$$

The use of Roman numerals persisted in Europe as late as the end of the sixteenth century. Only with the coming of the printing press did the Hindu-Arabic numerals which we now use begin to come into common use. During the latter part of the sixteenth century in Europe, a method for adding numbers was developed, apparently as an outgrowth of the use of the abacus. This method resembles the algorithm which we use today, except that the addition algorithm began with the left-most digit of the numerals and proceeded to the right. This caused no difficulty in an example like 42 + 35, but in an example like 47 + 35, difficulty arose. After the 4 and 3 were added, and a 7 recorded, the units digits were added. This addition gave 12, so the 2 was recorded in the units column, the 7 in the tens column was scratched, and an 8 was put below it. The answer was then represented by the final set of digits.

At first glance, this method does not seem to be much more trouble than the one we now use, but let us consider the first problem in Figure 5–1 which is outlined in its successive stages, with additions proceeding from left to right. In this problem we can see that the addition of the units column caused revision in the tens column of the sum already

Addition with Roman Numerals		
XX	V	II
X	V	IIII
XXX	VV	IIIII
XXX	VVV	I
XXXX	V	I

47
+ 35
7
47
+ 35
~~7~~2
8

shown there, so that this sum was "scratched." The sum in the hundreds column was then altered, so that it had to be "scratched," and, finally, the sum in the thousands column was altered and had to be changed. This obviously is more work than is necessary and illustrates why the algorithm which we use today (which starts with the units column) enables us to obtain results much more rapidly than our counterparts four hundred years ago could obtain them.

			Or consider the following problem.		
746	746	746	776	776	776
582	582	582	582	582	582
637	637	637	667	667	667
961	961	961	981	981	981
27	2~~7~~1	2~~7~~~~1~~6	27	2~~7~~9	~~2~~~~7~~~~9~~6
	9	92		9	3~~9~~0
					0
		2926			3006
		answer			answer

FIGURE 5–1

While it may be more sensible to deal with the largest and hence most important part of each number first (as was done in the "scratch" method just illustrated), increased speed of operation has dictated that we use the algorithm in common practice today. However, the very fact that numeral systems and algorithms have changed should cause us to approach the subject of addition with our minds open to different, new, and perhaps better ways to accomplish our additions.

Briefly, now, let us consider addition of fractions. Here too, historically, methods and techniques have changed. The ancient Egyptians

had an unwieldy system of fractions because they expressed every fractional part of a whole as the sum of fractions with unit numerators. That is, $\frac{7}{8}$ would have been written as $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$. This led to very tedious methods in the arithmetic of fractions. Today, we use a method of adding fractions which has been reduced, by the definition given earlier, to a problem of finding common denominators and then adding numerators.

II. MULTIPLICATION

Concepts to be developed in this section are:

1. *Multiplication may be defined in terms of repetitive addition or in terms of sets.*
2. *Algorithms are devised which conform to appropriate definitions of the process and the properties of arithmetic which are involved.*
3. *The multiplication algorithm we use today is only one of many algorithms which could be used.*
4. *Computers which follow the basic pattern of the algorithm have been devised to simplify the labor involved in computation.*

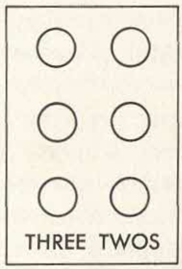
THE NATURE OF MULTIPLICATION

How many ways can we multiply 824 by 237? We are going to consider several ways in this section. We are also going to look at what multiplication is, how and why our algorithm for multiplication works, and how some ancient and other modern algorithms for multiplication work. We will also take a brief look at some devices for performing multiplications, but initially let us look at what multiplication is.

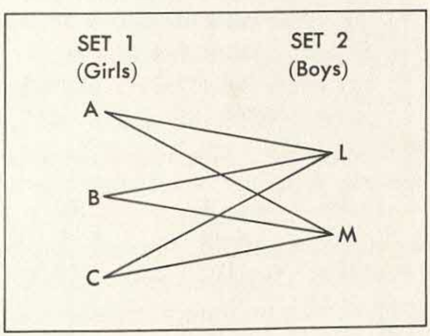
We should confine our attention first to multiplication of natural numbers. Take as an example 3×2. There are at least two different definitions of this product which we should consider, however, they both result in the same answer. One of these definitions is the one with which we are all familiar, and that is that $3 \times 2 = 2 + 2 + 2$. This definition is couched in terms of repetitive addition.

In the primary grades it may be preferable to read "3×2" as "three twos." Later, as the child becomes more familiar with multiplication as an abstract process, the reading might well be changed to "three times two."

The other definition is given in terms of sets. It is that the product 3×2 represents the number of ways in which the three members of one set (or group of objects) may be paired with the two members of some second set. As an illustration, at a school dance there are 3 girls and 2 boys. How many ways can the 3 girls and 2 boys be paired? This is diagrammed in the accompanying illustration. Boy *L* can be paired with girls *A*, *B*, and *C*, and boy *M* can be paired with girls *A*, *B*, and *C*.



It is easy to see that there are six possible pairs of elements. Of course, we might say that this is virtually equivalent to repetitive addition, and such is the case. However, a slightly different light is shed on the meaning of product in this latter sense, and the set theoretic approach is probably more modern in flavor.



It is desirable to point out that "product" in this sense (numerical product) is quite distinct from "product" in the sense of logical product. This latter type product is studied in connection with systems of logic, deductive reasoning, inference, etc. We will confine our attention here to numerical products since these are the ones encountered in the elementary school. Further, our attention in this chapter is directed to the products of natural numbers; products of other kinds of numbers are treated elsewhere.

MULTIPLICATION ALGORITHMS

When we write down the numerals representing natural numbers and begin to perform multiplications, digit by digit, we are applying certain fundamental properties. The properties we apply are the commutative property, the associative property, and the distributive property. These properties may be stated symbolically as follows:

Commutative Property of Multiplication: $ab = ba$

Associative Property of Multiplication: $abc = a(bc) = (ab)c$

Distributive Property of Multiplication with Respect to Addition: $a(b + c) = ab + ac$

Parenthetically, the commutative property also enables us to write the left member of the distributive property $(b + c)a$. The example in Figure 5–2 will serve to illustrate how we apply these properties in working a multiplication problem. When we multiply 24 by 37 by the standard algorithm, we are applying all three of these fundamental properties. First, we can say that we may use the commutative prop-

A

$$\begin{array}{r} 24 \\ \times\ 37 \\ \hline 168 \\ 72 \\ \hline 888 \end{array}$$

B

$$\begin{aligned} & 37 \times 24 \\ &= (30 + 7) \times (20 + 4) \\ &= [(30 + 7) \times 20] + [(30 + 7) \times 4] \\ &= (30 \times 20) + (7 \times 20) + \\ &\quad (30 \times 4) + (7 \times 4) \\ &= 600 + 140 + 120 + 28 \\ &= 888 \end{aligned}$$

C (1's × 1's) (10's × 1's) (1's × 10's) (10's × 10's)

$$\begin{array}{r} 4 \\ \times\ 7 \\ \hline 28 \end{array} + \begin{array}{r} 20 \\ \times\ 7 \\ \hline 140 \end{array} + \begin{array}{r} 4 \\ \times\ 30 \\ \hline 120 \end{array} + \begin{array}{r} 20 \\ \times\ 30 \\ \hline 600 \end{array} = 888$$

$$\begin{array}{r} 24 \\ 37 \\ \hline 28 \\ 140 \\ 120 \\ 600 \\ \hline 888 \end{array}$$

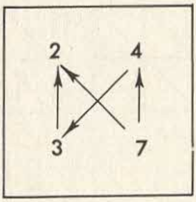
FIGURE 5–2

erty for multiplication, since it does not matter whether we multiply 24 by 37 or 37 by 24. Second, in the algorithm which we use, we literally treat the digits of the numerals separately. We are actually multiplying $20 + 4$ by $30 + 7$. In doing this we use the distributive property to enable us to multiply 20 by $30 + 7$ and 4 by $30 + 7$. (It might be commented that we also use the commutative property for addition here, but this has been discussed in another section.)

The associative property for multiplication does not find its way into our work until we multiply more than two numbers together. When this type problem confronts us, for example, in $3 \times 2 \times 7$, we use the associative property when we multiply any two of the factors together first.

In the example just given, that of 37×24, the analysis of the problem as $(30 + 7) \times (20 + 4)$ makes apparent why the algorithm we use for multiplication gives the correct answer. Thus, the second partial product results from 30×24 which is 720. In the algorithm we usually abbreviate this by eliminating the zero, as shown in A of Figure 5–2.

Following this pattern, we can develop a rule for writing down the product rapidly without putting in the intermediate steps. This "lightning" method requires that all intermediate steps of "carrying" be done mentally. It can be outlined graphically as follows: Multiply 7 times 4, tabulate the 8, "carry" the 2 (2 tens literally); multiply 4 times 3 (tens) and 7 times 2 (tens) obtaining 12 (tens) and 14 (tens) respectively; add the 12 (tens) and 14 (tens) and the 2 (tens) from the preceding "carry," obtaining 28 (tens); write down the 8 (tens) and "carry" the 2 (hundreds); multiply 3 (tens) times 2 (tens) and add the "carry" to get 8 (hundreds); write down this 8 (hundreds). The result is then 888.



This "lightning" method may also be applied to numerals of three or more digits, and although time and space preclude further discussion of this method it provides an opportunity for additional experiences in mental computation.

There are other modern algorithms which could be discussed, but let us turn our attention instead to some algorithms of more ancient origin and investigate them. Such an investigation reveals (a) the simplicity of the algorithm we use today, (b) the variety of algorithms which have been used in multiplication, and (c) the underlying principles inherent in the modern algorithm.

One of the oldest methods for performing multiplications is called the "lattice" or "grating" method. Let us look at Figure 5–3 for an

illustration of this method in the multiplication of 24 and 38. First, we construct a lattice as in diagram A. The number of blocks which appear as squares with diagonals across them is determined by the number of digits in the numerals to be multiplied. (If the numerals contained six and four digits, respectively, then the lattice would need to have dimensions of six blocks by four blocks.) We place the multiplier and the multiplicand as shown in B. Next, we multiply each digit of one numeral by each digit of the other, tabulating the results in the appropriate space. This is diagrammed in two steps, the first (C) to show one such multiplication, and the other (D) showing the entire multiplication.

Now, in the lattice part of the diagram, starting in the lower right hand corner, we add the numbers represented by the numerals along each diagonal carrying where necessary. In diagram E, arrows are shown to give directions for the additions, but they are omitted when we actually do such a multiplication. The answer is read counterclockwise around the left edge and bottom as 912.

LATTICE METHOD

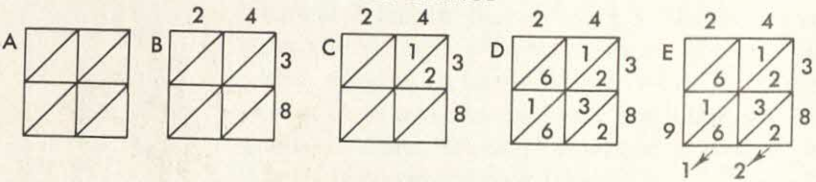


FIGURE 5–3

Based on the lattice method, an ingenious device was invented by John Napier around the year 1600. This device has been called "Napier's Bones." These "bones" consisted of pieces of wood on which successive multiples of a given number (representable by a single digit) appeared. When arranged in a special pattern with an index "bone," multiples of a number could be read. The bones for 2, 7, and the index are illustrated in Figure 5–4.

If we desired to multiply 27 by 6 by this method, we would put the bones side by side and read the answer by adding along diagonals the numbers represented by the numerals appearing opposite the index 6.

Multiplication of 27 by 53 would require reading two partial products and writing them in the proper sequence. This is done by first multiplying 3×27 and writing the product on the bones shown in A. Next, 50×27 is found by multiplying the product of 5×27 by 10, as

shown in B. These partial products are then added. Note the similarity between this and our modern algorithm.

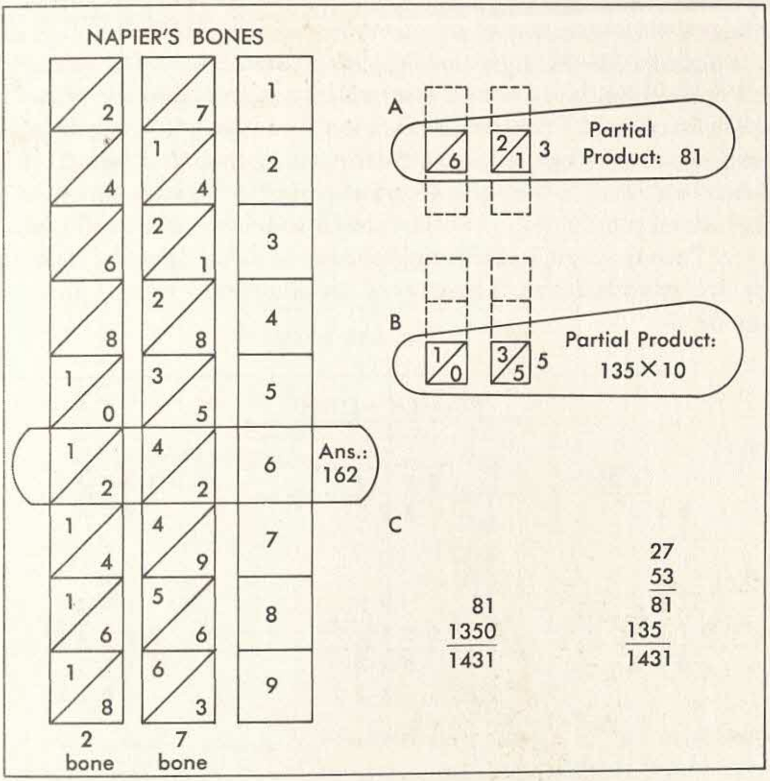


FIGURE 5–4

The people who used the lattice method during the sixteenth and seventeenth centuries accomplished by adding diagonally what we accomplish by shifting our partial products to the left in our algorithm. Of course, both our shift left and their diagonal addition are designed to keep the correct place value associated with each partial product.

Another algorithm which was used in Europe in the sixteenth century (and which had its counterpart, with slight modification, in India) is called the "scratch" method. In the "scratch" method, the work proceeds from left to right (as opposed to the right to left direction of the modern algorithm). If, for example, it is desired to multiply 47 by 823, the work is arranged as shown in Figure 5–5. A box is put around

the 47 to keep it separate from the remainder of the work. Next, each digit of 823 is multiplied by the digit 4 of the multiplier. This is in actuality a multiplication by 40, but the arrangement of the subsequent work guarantees that the place value of each product is maintained. After the product 4×8 is obtained, the 8 is scratched out, and above it (replacing it), is placed a 32—the digit 2 being placed above the 8. The next product 4×2 is found, the lower 2 is scratched out, and above it (replacing it), is written an 8. Then the product 4×3 is found. The product 12 is placed above the digit 4 of the multiplier so that the digit 2 of the product is directly above the 4, and the digit 1 is carried over to the next column causing the 8 in that column to be scratched and replaced by a 9. Then the digit 3 of the multiplicand and the digit 4 of the multiplier are scratched out. These steps are illustrated by the first four diagrams

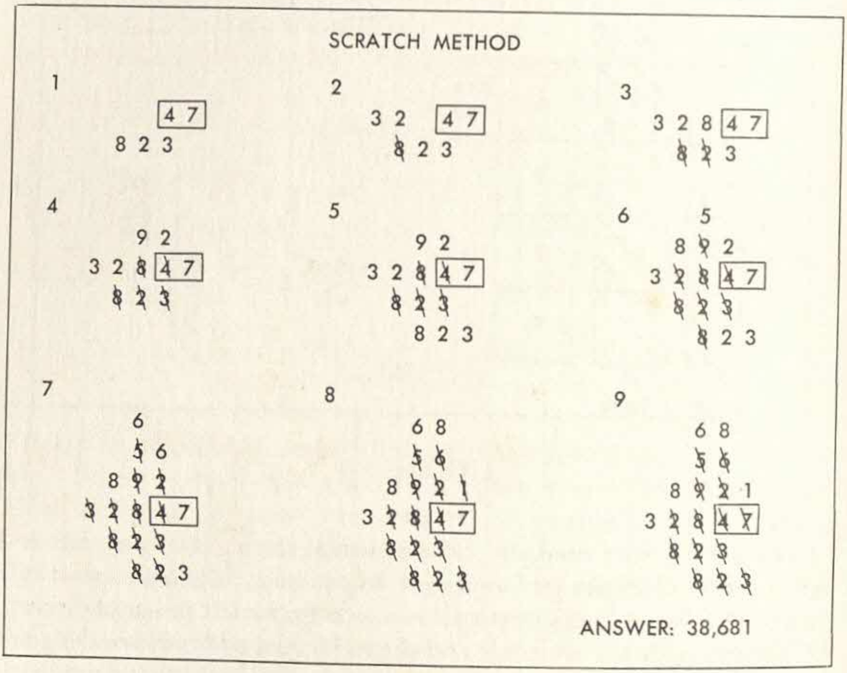


FIGURE 5–5

After these four steps are complete the multiplicand 823 is rewritten below its previous position and one place to the right. The whole process is then repeated. The digit 7 of the multiplier is multiplied by each of the digits of the multiplicand 823. As each successive product is

tabulated directly above the corresponding digit of the multiplicand, the preceding information in that column is scratched out, and new information is added to the old and written in above the old. If any "carrying" is necessary, it is also done by scratching out the old digit, adding to it the "carried" information, and writing down the new digit. These successive steps are shown in diagrams 5 through 9, Figure 5–5.

Another algorithm for multiplication is one called the Russian Peasant Method, illustrated in Figure 5–6. The multiplier and multiplicand are interchangeable here. One is doubled each time; the other is divided by 2. Halves are ignored in dividing. Lines with even numbers on the divide side are crossed out. The remaining partial products on the doubling side are added.

RUSSIAN PEASANT METHOD

24 x 26

÷2	×2
~~24~~	~~26~~
~~12~~	~~52~~
~~6~~	~~104~~
3	208
1	416
	624

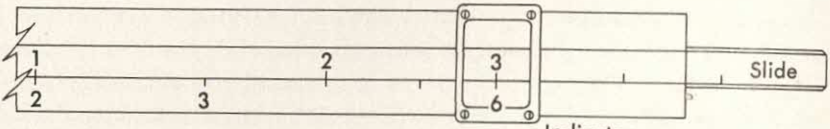
FIGURE 5–6

These various methods represent only a sample of the ways which have been devised for multiplying two natural numbers. The methods discussed and illustrated work equally well for multiplying two numbers which are expressed by numerals with bases other than ten.

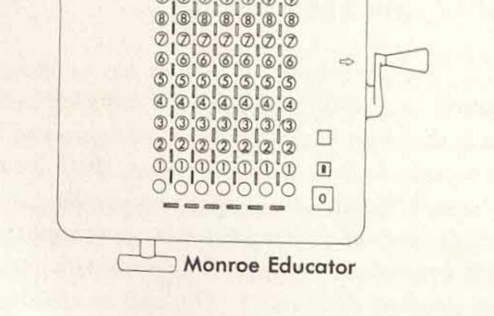
MULTIPLYING MECHANICALLY

An elementary mechanical device which has been invented to perform multiplication and other operations is the slide rule. In principle, its construction is based on logarithms. No discussion of logarithms will be given here because, at least for the present, they are not used in the elementary grades. The slide rule converts number into length, the length being dependent on the number but not proportional to it, and obtains products by adding lengths. The slide rule in the illustration is set to find the product of 2 and 3. The end of the slide is set at 2 on

the bottom scale, and the indicator is set at the 3 on the slide. The product 6 is read on the bottom scale. (For quotients, it subtracts lengths.) The modern engineer would be hard pressed to get all of his work done without this very simple and time-saving device.



Other mechanical devices which have been devised to perform multiplications include desk calculators and modern high-speed electronic computers. Both of these types of machines perform multiplications by repeated additions. There are certain modifications from machine to machine (such as the base used in representing the numerals on which the operations are performed), but they all employ a shifting technique as do each of the various multiplication algorithms. An early computer, the IBM 650 electronic data processing machine, multiplies the multiplicand by the leftmost digit of the multiplier through repeated additions of the multiplicand. It then shifts this product one place to the left, multiplies by the next digit of the multiplier, etc., until all digits of the multiplier have been used. This method of working from left to right on the digits of the multiplier resembles the "scratch" method of multiplication. (It might be of interest to note that the IBM 650, which was only an intermediate-speed computer, could multiply two numbers, each represented by ten digit numerals, in approximately 1/100



Monroe Educator

of a second, while late model computers can perform such a multiplication in approximately 1/1,000,000 of a second.)

The Friden desk calculator also performs multiplications automatically, but it reverses the order of the work. It performs repetitive additions, but it begins multiplying by the units (or ones) digit of the multiplier and works toward the left. In this regard its operation resembles that of our modern multiplication algorithm.

These methods can be demonstrated on a hand-operated desk calculator, such as the Monroe Educator. These small calculators are being used in a number of elementary school classrooms to enhance the child's understanding of our basic algorithms.

III. SUBTRACTION AND DIVISION

Concepts to be developed in this section are:

1. *Subtraction and division are secondary operations, defined in terms of addition and multiplication.*
2. *By definitions we expand our number system.*
3. *Three kinds of problems in the elementary school classroom involve subtraction, and it is frequently said that these give rise to three kinds of subtraction.*
4. *Two kinds of problems involve division, and it is frequently said that these give rise to two kinds of division.*
5. *As is true of the two primary operations, many algorithms have been developed to perform subtractions and divisions.*

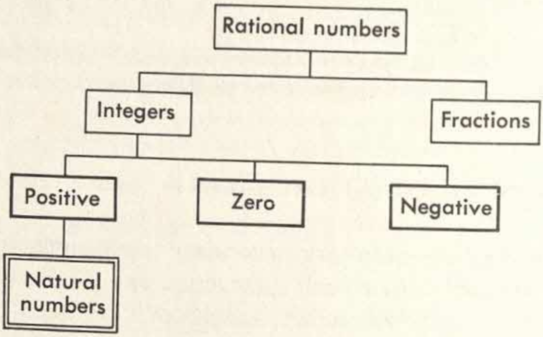
DEFINITION OF SUBTRACTION

Subtraction and division are secondary operations in arithmetic. Since it may be confusing to call subtraction and division both "fundamental processes" and "secondary operations," we need to distinguish between these two terms. Fundamental processes refer to the four arithmetic processes—addition, multiplication, subtraction, and division—which we must be able to perform in order to assume our places in society. These processes are fundamental in terms of our needs.

The term "secondary operations" is used to indicate that subtraction and division are derived from the primary operations of addition and multiplication. Before we can define subtraction and division in terms of sets, we must first define addition and multiplication. In the sense

that they are defined second—that is, in terms of other operations—subtraction and division are secondary.

How is subtraction defined? We must first say that we will use the minus sign to indicate that subtraction is to be performed. Next, we must agree that we will work only with numbers for which addition is defined. Now we say that $a - b = c$ (that b subtracted from a gives an answer c) *if* there exists a number c for which $a = b + c$. With one stroke we have defined subtraction for all kinds of numbers for which we have defined addition—*except* that there may be no such number c among the numbers for which we have defined addition. For example, if we have defined addition only for the natural numbers and for the rational numbers which can be formed by the quotient of two natural numbers (we will consider quotients in general later in this section), then $3 - 7$ is meaningless. A natural number does not exist which when added to 7 will give 3. Nor would $3 - 3$ have any meaning, for no natural number exists for which $3 + N = 3$. It is this very fact which caused negative numbers and zero to be included in our number system. If negative numbers and zero are included in our number system, though, we then have to go back and re-define addition and multiplication for these numbers. Having defined $7 + (-4)$ to mean $7 - 4$, which is 3, we then can say that the problem $3 - 7$ has meaning, and that the answer to the problem is -4.

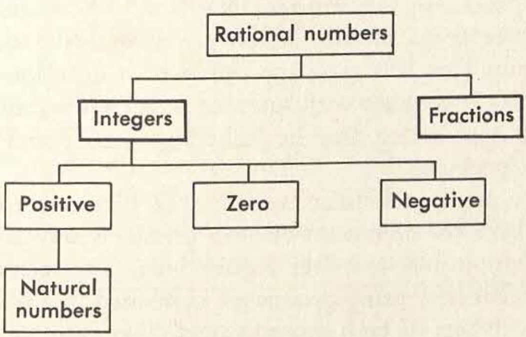


Historically, it was quite a long time after such an approach to subtraction and negative numbers was proposed that it gained any wide acceptance. In fact, it was not until the sixteenth century that negative numbers were finally admitted to the households of proper mathematicians. We might take great pride in our achievements when we read that the medieval mathematician sneered at negative numbers as being meaningless if it were not for the fact that some people even today maintain the same attitude.

With the definition given for subtraction, we see that we can pair any two numbers (for which addition is defined) in the subtraction operation and that the result of the subtraction will also be a member of our set of numbers. This ability to pair any two members in this operation is called the principle of completeness. The principle of closure is demonstrated by the fact that any subtraction operation yields a result that is also a number in the system.

DEFINITION OF DIVISION

We define division in terms of multiplication. When we speak of the quotient of a divided by b or merely a divided by b, we mean the number c (if one exists) for which $bc = a$. This is stated in symbols $a \div b = c$ *if* there is a number c for which $a = bc$. We can see almost at once that if our number system is limited to the natural numbers, then we may not be able to divide some numbers into some other numbers. For example, while $6 \div 3$ would have meaning among the natural numbers since there does exist a natural number 2 for which $6 = 3 \times 2$, if the problem were $7 \div 3$ there could be no answer since no natural number exists for which $7 = 3 \times N$. This condition caused the introduction of still more numbers into the number system. Whereas multiplication and addition could always be performed on natural numbers, subtraction and division could not. Because we always wanted to have a result in subtraction, we introducted negative numbers and zero into the number system; and because we wanted to insure that division also would always give an answer within the system, we introduced fractions (both positive and negative) into the number system. There is one reservation that must be made in this introduction of new numbers by division, and that is that division by zero must be excluded. That is, not all pairs of numbers may be related in the divi-



sion operation, or, said more technically, the operation of division is not complete. If we approach the problem of $7 \div 0$ from the standpoint of the definition of division, we are asking that a number N be found so that $0 \cdot N = 7$. Since this cannot be done in our number system, division by zero is excluded.

Division of natural numbers may also be defined in terms of repeated subtractions, just as multiplication can be defined in terms of repeated additions. We will see that our modern algorithms for performing divisions are made up of a little of both possible ways of defining division. It might be of interest, too, to note that modern calculators and computers do perform divisions by a combination of shifting and repetitive subtractions in much the same way our algorithms operate.

KINDS OF PROBLEMS INVOLVING SUBTRACTION AND DIVISION

Leaving behind the definitions of subtraction and division, let us move on to the types of subtraction problems which occur in the classroom. These are problems which ask: (a) How many are left? (b) What is the difference? and (c) How many more are needed? (See Figure 5–7.) We can recognize the first of these as the "take away" problem of the primary grades. The second—what is the difference?—is a type of subtraction problem in which only the numerical value of the difference is desired. This type of problem is one of placing members of one group into one-to-one correspondence with members of another group, and then finding how many members of one of the groups have no partner in the other group. Such a subtraction as this is involved when a child says, "But their team has three more people on it than ours." We can recognize the third type of subtraction problem—how many more are needed?—as the definition of subtraction given earlier, that $7 - 3 = N$ means $N + 3 = 7$. Teachers who recognize these three types of subtraction problems make certain that in work with pupils each is used appropriately. Confusion often results when children's experience with subtraction is confined to a single type of operation. Subtraction may be "take away" or either of the other two types of problems.

Elementary pupils encounter two types of division problems. These are measurement (or quotitive) division problems and partitioning (or partitive) division problems (see Figure 5–8). The measurement type of problem seeks how many groups are contained in a whole when the number of members of each group is fixed. For example, "How many

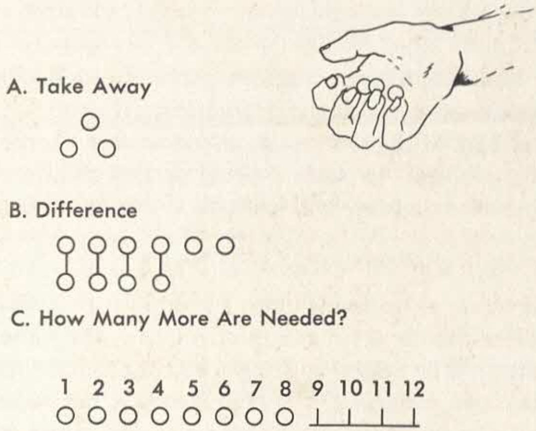


FIGURE 5–7

teams of four children each can be formed from a group of 12 children?" In this type of problem (reverting to the definition of division) the multiplier is being sought, and the units of the dividend and divisor

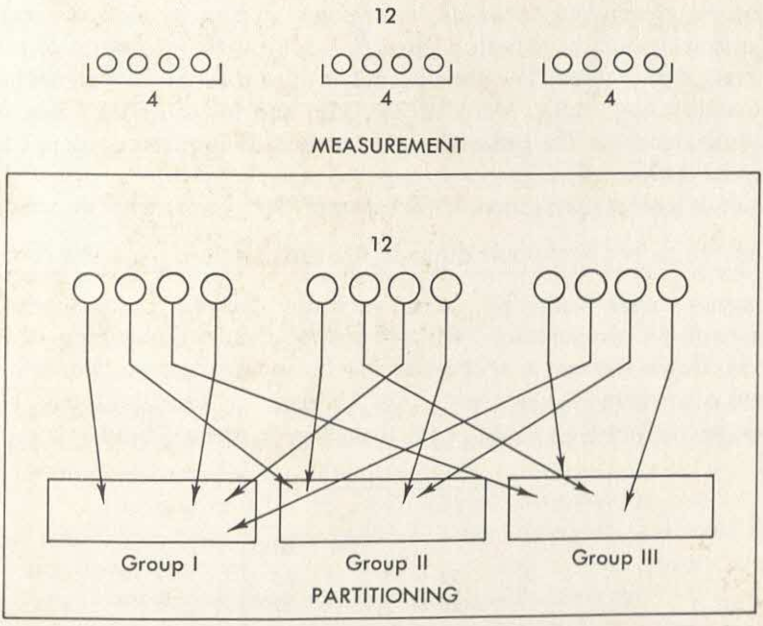


FIGURE 5–8

(children) are the same. This can be shown as $\frac{12 \text{ children}}{4 \text{ children}} = 3$, in which the quotient 3, although it may indicate groups, does not have the same set of units possessed by both the divisor and dividend.

The second kind of division problem encountered is the partitioning (or partitive) problem. In such a problem the child is to find how many members each group will contain if the number of groups is fixed. This may be illustrated as before: $\frac{12 \text{ children}}{3} = 4$ children. That is, if 12 children are to be divided into 3 groups, there will be 4 children per group. Here the dividend and quotient have the same set of units.

While it may not be necessary for the teacher to have memorized the names of these two processes, it is important for her to recognize that these two types of problems do occur in concrete situations. Children's understanding may be enhanced through use of experiences which emphasize diagrams and models of problem situations.

SUBTRACTION AND DIVISION ALGORITHMS

Before concluding, it would be of some interest to look at several algorithms which have been devised to perform the operations of subtraction and division. We are all familiar with the standard algorithms commonly used today. Many others exist, and for some purposes, for example, teaching the reasons that the standard algorithms work, they may be useful.

Let us look at subtraction. If we subtract 47 from 65, what do we do? First, we write the example down in standard fashion, $\begin{array}{r} 65 \\ -47 \\ \hline \end{array}$. We cannot subtract 7 from 5 directly, so we exchange one ten for ten ones. In this form we can subtract 7 from 15 and 40 from 50, obtaining $10 + 8$ or 18. Pupils sometimes abbreviate this by saying that we "borrow 1" from 6 to make 15, but the "1" we "borrow" is literally 1 ten. This may be seen easily by use of an abacus with twenty beads on a rod

	51
50 + 15	65
−(40 + 7)	47
	18

(see Figure 5–9). First, we set the problem on the rods by putting 6 beads on the tens rod and 5 beads on the units rod to indicate 65. We then try to take 4 beads away from the number of beads on the tens rod and 7 beads away from the number on the units rod (A). Since we cannot take 7 beads from 5 beads, we exchange 1 bead on the tens rod for ten beads on the units rod. The abacus will then appear as in B. Now we take 7 beads from those on the units rod and 4 beads from those on the tens rod, leaving an answer which is interpreted to be 1 ten and 8 ones or 18 (C). We readily see that the 1 which was borrowed was 1 ten.

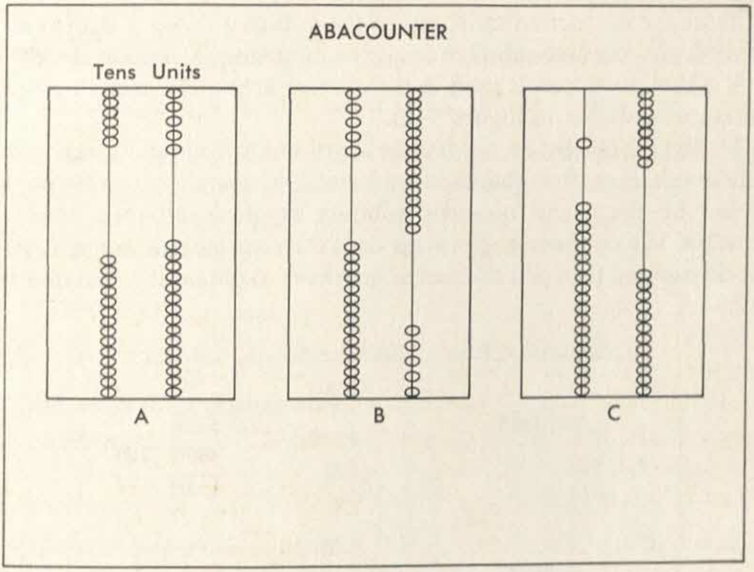


FIGURE 5–9

An interesting scheme has been devised to avoid borrowing in subtraction. It is based on the definition of subtraction and on the commutative property of addition. The scheme is:

$$65 - 47 = (65 + 3) - (47 + 3) = 68 - 50 = 18.$$

This could have been done equally well by adding 4 to each of the numbers involved:

$$65 - 47 = (65 + 4) - (47 + 4) = 69 - 51 = 18.$$

In neither case was "borrowing" or "exchanging" in the ordinary sense used.

An extension of this equal additions method is in common use in many places today. The equal addition, as illustrated at the right, is usually one ten in the tens column of the subtrahend and ten ones in the ones column of the minuend.

$$
\begin{array}{r}
65 \\
-47 \\
\hline
\end{array}
\qquad
\begin{array}{r}
(60 + 5) \\
-(40 + 7) \\
\hline
\end{array}
\qquad
\begin{array}{r}
(60 + 15) \\
-(50 + 7) \\
\hline
(10 + 8) \\
\\
18
\end{array}
$$

There are instructive variations of the ordinary division algorithm. A method of successive subtractions is the most simple division algorithm.

A technique which is used in the current arithmetic series by Scott, Foresman is shown in Figure 5–10.

To divide 5088 by 24, we lay the work out as follows. Twenty-four will divide into five thousand at least 200 times. We multiply the divisor by the partial quotient, subtract as usual and then divide 24 into 288. We continue the process until the remainder is not as large as the divisor and then add the partial quotients to obtain the final quotient 212.[3]

A

```
24 ) 5088 |
          |
```

B

```
24 ) 5088 |
     4800 |  200
     ----
      288 |
```

C

```
24 ) 5088 |
     4800 |  200
     ----
      288 |
      240 |   10
      ---
       48 |
```

D

```
24 ) 5088 |
     4800 |  200
     ----
      288 |
      240 |   10
      ---
       48 |
       48 |    2
     -----------
             212
```

FIGURE 5–10

[3] Maurice L. Hartung, Henry Van Engen, and Lois Knowles, *Seeing Through Arithmetic 4* (Glenview, Illinois: Scott, Foresman & Company, 1956).

The above approach, while excessively long in some cases, illustrates exactly how our usual algorithm works. If only the single non-zero digits of the partial quotients are written, and if the divisor is not recopied at every step, the lengthier approach very closely resembles the common division algorithm.

There are many other points of interest in the study of subtraction and division, but each one of us should have found enough material in what has already been given to start us off in search of new facts about and new insights into the subjects of subtraction and division.

EXERCISES

1. By how much does the sum of 25 and 17 exceed their difference?

2. By how much does the product of 24 and 19 exceed their sum?

3. Turn ruled paper so that the lines are vertical. Use these vertical rulings as guides for maintaining place value in working each of these problems:
   (a) $534 + 322 + 27 + 2169$
   (b) $5346 - 728$
   (c) 322×27
   (d) $5346 \div 27$

4. Discuss the merits of the vertical ruling used in Problem 3.

5. Add, using the scratch method.

   (a) $\begin{array}{r} 789 \\ 673 \\ \underline{548} \end{array}$ (b) $\begin{array}{r} 329 \\ 762 \\ \underline{147} \end{array}$ (c) $\begin{array}{r} 652 \\ 128 \\ \underline{543} \end{array}$

6. Subtract, using equal additions.

   (a) $\begin{array}{r} 82 \\ \underline{-46} \end{array}$ (b) $\begin{array}{r} 126 \\ \underline{-\ 98} \end{array}$

7. Multiply, using the lattice method.

   (a) $\begin{array}{r} 53 \\ \underline{\times 64} \end{array}$ (b) $\begin{array}{r} 472 \\ \underline{\times\ 93} \end{array}$

8. Use the Russian peasant method to multiply 57×62.

9. Multiply the two exercises in Problem 7 using the conventional algorithm.

10. Multiply 238 by 97 using the scratch method.

11. Discuss a definition of multiplication of the natural numbers a and b as

the number of elements in a rectangular array whose dimensions are a by b.

12. Show how the division $84 \div 7$ can be illustrated by use of arrays. Let your illustration also show the distributive property:

$$84 \div 7 = (70 \div 7) + (14 \div 7)$$

13. Show how an array of 7 rows of 12 dots each can be made to illustrate
    (a) the product of 7×12.
    (b) the commutative property of multiplication.
    (c) partition division.
    (d) measurement division.

14. Arrange 52 dots into rows of 7 dots each. Explain how this arrangement of dots can be used to illustrate division with a remainder.

15. Write a summary of the rules for adding and subtracting both positive and negative numbers.

16. Describe a method for dividing natural numbers using only repeated subtractions.

17. Find and report on a method for finding the greatest common divisor of two natural numbers.

18. Investigate how addition and multiplication were done using Roman numerals.

19. Discuss the statement: "Addition is the basic process with other processes defined in terms of addition."

20. Find and report on algorithms (other than those discussed in this chapter) for addition, subtraction, multiplication and division.

21. Show the relationship between our multiplication algorithm and the lattice method.

22. If we restrict the subtraction process to the natural numbers, what limitations must we place on the definition $a - b = c$? Explain.
    (a) $a \neq b$, $a = b + c$
    (b) $a \neq c$, $a = b + c$
    (c) $a < c$, $a = b + c$
    (d) $a > b$, $a = b + c$

23. Which of the following is a method the teacher may use to emphasize that subtraction does not exist logically until after addition has been defined?
    (a) teach the addition and subtraction facts together.
    (b) teach the subtraction algorithm immediately after teaching the addition algorithm.

(c) define addition in terms of the subtraction algorithm.
(d) limit the definition of the addition algorithm to natural numbers and define the subtraction algorithm for integers.

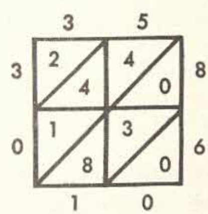
24. Which of the following is a method the teacher may use to emphasize that division does not exist logically until after multiplication has been defined?

(a) teach the multiplication and division facts together.
(b) teach the division algorithm immediately after teaching the multiplication algorithm.
(c) define multiplication in terms of the division algorithm.
(d) limit the definition of the multiplication algorithm to natural numbers and define the division algorithm for integers.

Select the appropriate response to each of the following statements and explain the reason for your choice.

25. In which of these algorithms for multiplying 35 by 86 is the final step not the addition of all partial products?

(a)



(b)

÷2	× 2
35	86
17	172
~~8~~	~~344~~
~~4~~	~~688~~
~~2~~	~~1376~~
1	2752
	3010

(c)

$$\begin{array}{r} 35 \\ \times\ 86 \\ \hline 210 \\ 280 \\ \hline 3010 \end{array}$$

26. Teachers can use the lattice method of multiplication in:

(a) providing practice with the multiplication facts;
(b) helping develop an understanding of the historical development of mathematics;
(c) providing practice in the addition of partial products;
(d) helping develop an understanding of place value.

27. The requirements for use of the scratch method for addition vary from those for use of the standard algorithm in which of the following ways?

(a) knowledge of addition facts.
(b) carrying.
(c) adding from right to left.
(d) none of the above.

Extended Activities

1. Express the following in Roman numerals, then add:

(a)	(b)	(c)	(d)	(e)
57	81	97	5482	658
34	49	63	4309	391

2. Work Extended Activity 1 as if the exercises were subtraction exercises.

3. Discuss the following division algorithm, explaining the role played by each partial quotient, where and how the entire quotient is found, and the maintenance of place value by such an algorithm.

```
     63
    ---
      3
     60
93)5859
   5580
   ----
    279
    279
    ---
```

4. Show how the algorithm in Extended Activity 3 compares with the multiplication algorithm:

```
  63
  93
 ---
 189
5670
----
5859
```

5. Some teachers find that children in their classrooms have learned a long division algorithm taught in European schools. In this algorithm only the remainders are written.

   Show what sequence of mental steps in working the example in Extended Activity 3 could lead to the following steps being written (the first of which is an integral part of the second):

```
      6                63
93)5859           93)5859
    279                279
```

6. Use the method illustrated in Extended Activity 5 to perform the division $3826 \div 27$.

7. Distinguish between two kinds of remainders in division—whole and fractional—and their roles.

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Chapter 6

Concepts to be developed in this section are:

1. *Many sets of numbers can be identified, and questions concerning the relevance of certain properties to each set can be raised.*
2. *Much of what mathematics is can be seen in the study of the relationship of properties to various sets of numbers.*
3. *Understandings related to prime factors, greatest common divisors, and least common multiples are needed in work with fractions.*

6

Identifying Classes of Numbers

EXPLORING KINDS OF NUMBERS

In the study of arithmetic, sets of numbers are identified and various explorations are made with these numbers to reveal certain properties which hold for these numbers under specific conditions. In previous chapters, we have learned that for the set of whole numbers the property of associativity holds for the operation of addition. That is, if a, b, and c represent numbers from the set of whole numbers, then the order of grouping of these as addends in addition does not change their sum $[(a+b)+c=a+(b+c)]$. For instance, if $a=2$, $b=3$, and $c=4$, then $(2+3)+4=2+(3+4)$. Other questions which we asked about the set of whole numbers included commutativity under the operation of addition, existence of identity elements, closure, etc. These and other questions were asked about this set of whole numbers under the operation of multiplication. Much of what we call number theory involves just this kind of exploration in arithmetic.

Odd and Even Numbers

In the early years of the elementary school after children have developed some understanding of the set of whole numbers, they begin to recognize two kinds of numbers. In their explorations with objects, elements of sets, they see that for some sets it is possible to match elements into pairs and no single element will be left over. For other numbers, however, they see that when elements are paired, one is left over. These children begin to recognize that numbers can be divided between those which are even and those which are odd. In the middle grades, these children may learn that an even number is one which has as one of its factors, 2. Thus, $2n$ could be used to designate any given even number in which 2 times some whole number n gives the even number e ($2n = e$). An odd number, on the other hand, when divided by two will leave a remainder of 1, and so an odd number may be designated as $2n + 1 = o$.

Let us consider, then, some questions concerning the set of even numbers. Is the set of even numbers closed with respect to addition? This question simply asks if when any two even numbers are added, will the sum also be in the set of even numbers—will the sum be an even number. The several examples in Figure 6–1 each indicate that the sum of two even numbers is an even number.

4 + 2 = 6	26 + 14 = 40	126 + 42 = 168
6 + 8 = 14	22 + 76 = 98	342 + 126 = 468
2 + 10 = 12	40 + 16 = 56	460 + 678 = 1138

FIGURE 6–1

In each instance we have examined, an even number added to an even number resulted in a sum which was another even number. We could continue our list of illustrations searching for a single example which would prove for us that such is not always the case. It is impossible, however, for us to look at all possible examples and so we shall try to answer the question in another way. If $2a$ and $2b$ represent any two even numbers, then our question asks whether $2a + 2b = e$, where e is an even number. By the distributive property we can show that $2a + 2b = 2\ (a + b)$. Because 2 times any number gives an even number, we are assured that $2a + 2b$, or the sum of any two even numbers, is also an even number. Is addition associative within the set of even numbers?

If d, e, and f are from the set of even numbers, is addition associative? Is $(d + e) + f = d + (e + f)$? Because we know that any number we choose for d, e, or f is an even number and therefore is a member of the set of counting numbers; and because we know that the counting numbers are associative under addition, it follows that any subset of the counting numbers is also associative under addition. Therefore, the even numbers are, indeed, associative under addition. Likewise, we could show that the set of even numbers is commutative under addition. That is, $(j + k) = (k + j)$. We have shown that the set of even numbers under addition is closed, associative, and commutative.

Let us turn our attention briefly to the set of odd numbers. Is the set of odd numbers closed with respect to addition? If q and r are odd numbers, is $q + r = o$, where o is another odd number? If $q = 3$ and $r = 5$, then $q + r = 3 + 5 = 8$, and in our first example we find that our set of odd numbers is not closed in this instance and we need but one false illustration to prove that the closure property does not hold. We could continue our exploration of this property, however, and show why this property does not hold for addition. In Figure 6–2, the definition of one odd number as $2n + 1$ and of another as $2m + 1$ is used. In the final step $2(m + n)$ is an even number and 2 is an even number. We have seen above that two even numbers when added give an even number (closure property). Thus we see that the addition of two odd numbers results in not an odd number but an even number and that the set of odd numbers, therefore, is not closed under addition.

$$\begin{aligned}(2n + 1) + (2m + 1) &= 2n + (1 + 2m) + 1 \\ &= 2n + (2m + 1) + 1 = (2n + 2m) + (1 + 1) \\ &= 2(n + m) + 2\end{aligned}$$

FIGURE 6–2

Following the reasoning which was established earlier concerning the associative and commutative properties of the even numbers under addition, do you expect the odd numbers to be associative and commutative under addition?

Figurate Numbers

Another interesting way to organize sub-sets of numbers within the set of counting numbers is to look at the geometric shapes in which

they might be conceptualized. Triangular numbers have interested mathematicians for many centuries. Triangular and square numbers are illustrated in Figure 6–3.

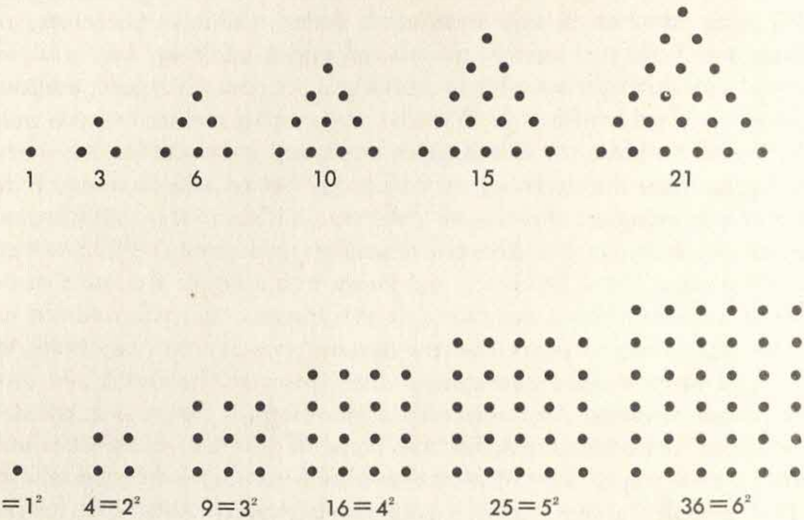


FIGURE 6–3

There are several interesting aspects of these patterns for figurate numbers. The sum of consecutive counting numbers starting with one (1) will be a triangular number.

$1 = 1$

$1 + 2 = 3$

$1 + 2 + 3 = 6$

$1 + 2 + 3 + 4 = 10$

$1 + 2 + 3 + 4 + 5 = 15$

$1 + 2 + 3 + 4 + 5 + 6 = 21$

$1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$

Interestingly, the square numbers are the sums of two consecutive triangular numbers, and the reason why this is so is illustrated in Figure 6–4.

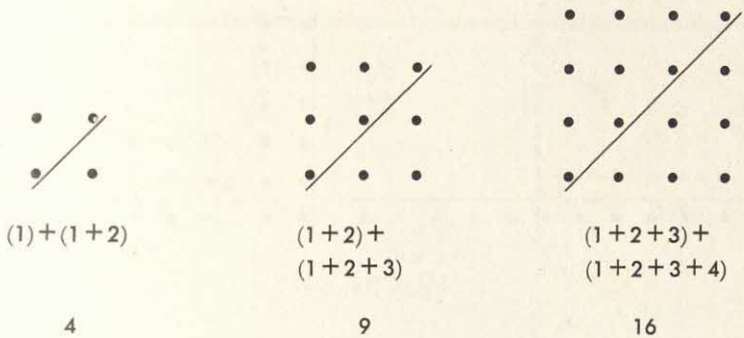


FIGURE 6–4

Any mathematical activity which provides opportunities for the child to make explorations into the realm of number is a useful, practical activity because it contributes to his interest (which is a major objective of the elementary mathematics program) and provides him with the tools as well as the incentive for further explorations.

New mathematics programs make particular use of rectangular numbers as a way of thinking about multiplication. The two factors of a given number may be considered the length and width of a rectangle. The visualizations of the rectangular numbers are called arrays and several are illustrated in Figure 6–5.

With each triangular number and with each square number there is only one geometric shape which can represent that number as a triangle or as a square. A single rectangular number, on the other hand,

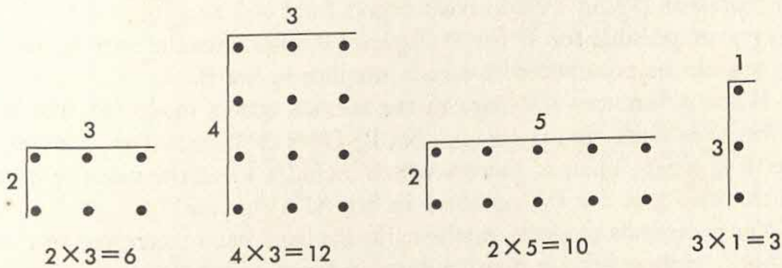


FIGURE 6–5

may be represented by a variety of arrays as illustrated for the number 12 in Figure 6–6.

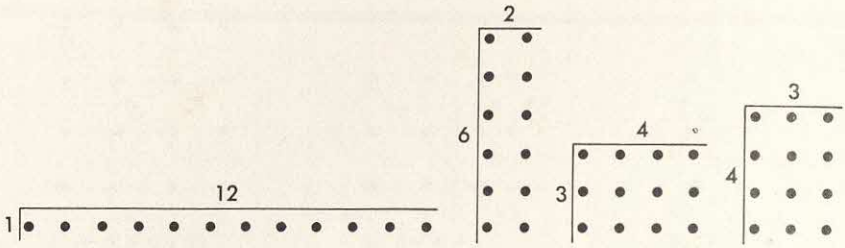


FIGURE 6–6

Prime and Composite Numbers

Given two sets of numbers, Set A and Set B, consider the kinds of rectangular arrays which can be used to illustrate the numbers of each set.

Set A = {4, 6, 9, 12, 18}

Set B = {2, 3, 5, 7, 11}

Consider first the numbers in Set A. What arrays can we construct to represent 4? Can we construct arrays for each of the following pairs of products: 1×4, 4×1, 2×2? What arrays are possible for 6? for 12? Figure 6–7 illustrates the various arrays which can be constructed for each number in Set A.

Now, consider the numbers in Set B. What arrays can we construct to represent 2? Can we construct arrays for 1×2 and for 2×1? What arrays are possible for 3? for 5? Figure 6–8 illustrates the various arrays which can be constructed for each number in Set B.

What differences are there in the sets of arrays made for numbers in Set A as compared to those in Set B? Does each array for numbers in Set B illustrate a pair of factors which includes 1 and the number itself? Is this also true for the numbers in Set A? Why not?

For thousands of years, mathematicians have been interested in those whole numbers greater than 1 which, in terms of whole number factors can be expressed only as the product of 1 and the number itself. These

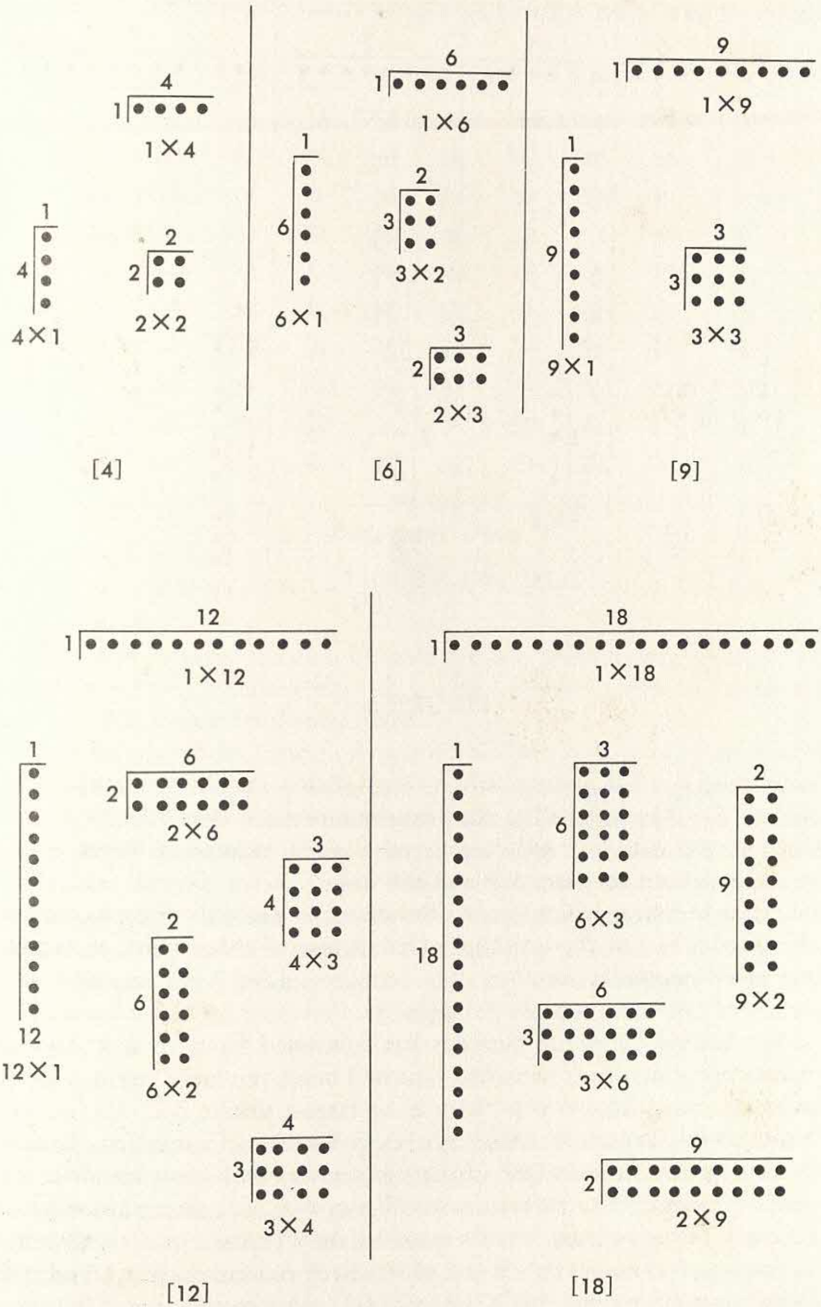

1×4
4×1
2×2
[4]
1×6
6×1
3×2
2×3
[6]
1×9
9×1
3×3
[9]
1×12
12×1
2×6
4×3
6×2
3×4
[12]
1×18
18×1
6×3
9×2
3×6
2×9
[18]

FIGURE 6–7

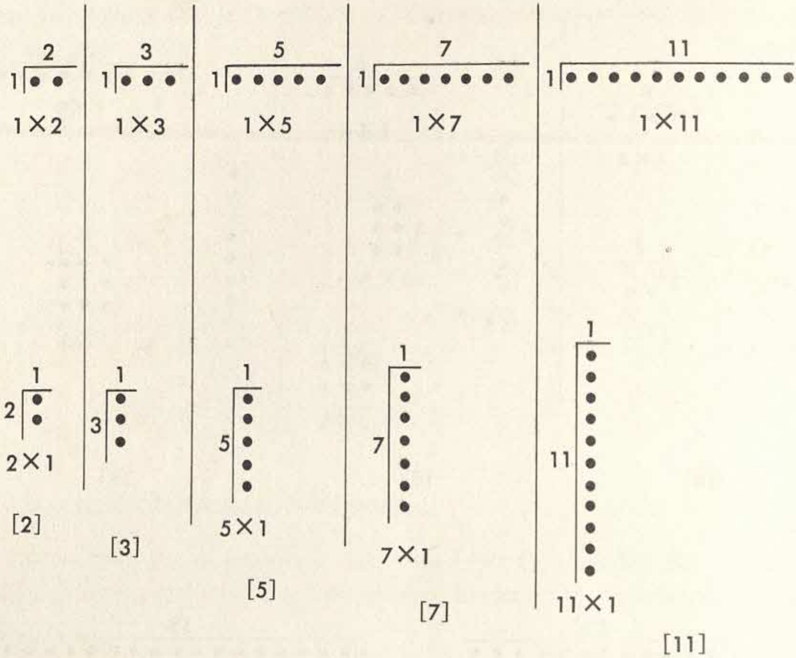


FIGURE 6–8

numbers are called *prime numbers*. Set B then is a set of prime numbers. In fact, Set B includes all of the prime numbers less than 12. (By definition the number 1 is not considered a prime number.) What is the next prime number after 11? and the next?

Those numbers which can be shown to be not only the product of the number 1 and the number itself but also of other pairs of factors are called *composite numbers*. The numbers in Set A are all composite numbers.

The search for prime numbers has continued from ancient days to the present time. Ever since the time of Euclid (circa 300 B.C.), mathematicians have known that there is no largest prime, but they do not know which large numbers are primes. A Greek mathematician, Eratosthenes, initiated the earliest systematic search for primes. He divised a grid of numbers like that shown in Figure 6–9. The first number, two (2), is a prime because it is the product only of itself and 1. All other multiples of 2, however, are the products of more than just 1 and that number; they are also the products of 2 and some number. Thus we cross out every multiple of 2—every second number—after the prime

number 2. (These have been crossed out in the table with a vertical mark.)

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Eratosthenes' Sieve

FIGURE 6–9

The next number is 3 and, because it is the product only of itself and 1, it is prime. All multiples of 3 after 3—every third number—are crossed off with a horizontal mark.

The number 4 is already crossed out so we know it is a multiple of some prime and is not itself a prime. The number 5 is a prime and all multiples of 5 after 5 are crossed out with an ×. Multiples of 7 following the prime number 7 are crossed out with a circle. All multiples of 11 are already crossed out. Why do we need to look no further before saying that those numbers not now crossed out are primes? We know that any number less than 11 (other than 1) times 11 is not prime by definition. The product of 11×11 is more than 100, so in our grid of 100 numbers we have now identified the prime numbers.

Prime Factors

It is sometimes useful to express composite numbers as the product of prime factors. The number 12 may be expressed as the product of 3 and 4. The number 4, however, is not prime and can itself be expressed as the product of two factors which are prime; i.e., 2 and 2. Thus, the prime factors of 12 are 3, 2, and 2; 12 may be expressed as $3 \times 2 \times 2$ (see Figure 6–10). Is there any different set of prime factors whose product is 12?

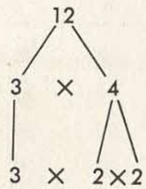


FIGURE 6–10

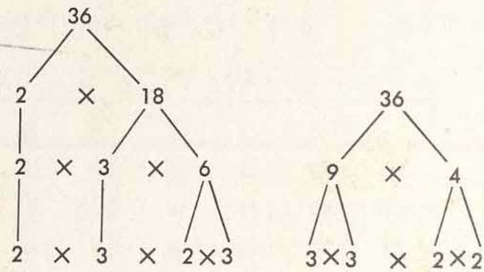


FIGURE 6–11

Prime factorizations are sometimes developed through factor trees. Figure 6–11 illustrates two ways in which the number 36 may be factored.

It should be noted that although the order of factors varies in each example, the sets of prime factors include the same four numbers. Figure 6–12 includes several patterns for factoring the number 24. In each instance does the set of prime factors include the same set of numbers?

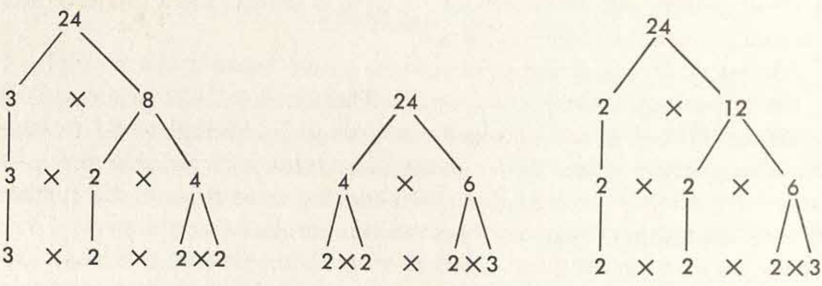


FIGURE 6–12

The fact that every composite number can be expressed as the product of the members of only one set of prime numbers is called the Fundamental Theorem of Arithmetic.

Because we frequently need to express numbers as products of prime factors, we should be able to determine as efficiently as possible the prime factors of any given number. This may be accomplished through a series of successive divisions. The number 84 can be seen to be the product of prime factors 2, 2, 3, and 7; the number 2730 has the prime factors of 5, 2, 3, 7, and 13.

2) 84	5) 2730
2) 42	2) 546
3) 21	3) 273
7	7) 91
	13

FIGURE 6–13

While the prime factors of 2730 have been found by successive divisions by primes, it should be apparent that an orderly consideration of possible prime divisors would make the determination more efficient. Thus, the following display is a better display of efficiency in finding prime factors.

2) 2730
3) 1365
5) 455
7) 91
13

Greatest Common Factor

Prime factorization is useful as we seek to determine the greatest common factor (GCF) of two or more numbers. Suppose we wish to determine the greatest common factor of 12 and 18. Set A includes the prime factors of 12, and Set B those of 18.

Set A = {2, 2, 3}
Set B = {2, 3, 3}

The prime numbers 2 and 3 are common to both sets, and so the product of 2 and 3 (or 6) is the greatest common factor of the numbers 12 and 18. This means that both 12 and 18 are divisible by 6 and that no number larger than 6 is a common factor of both 12 and 18. These common factors can be shown also with the use of Venn Diagrams.

The greatest common factor may be illustrated either with prime factors or with all factors for the numbers 36 and 48. Set A includes

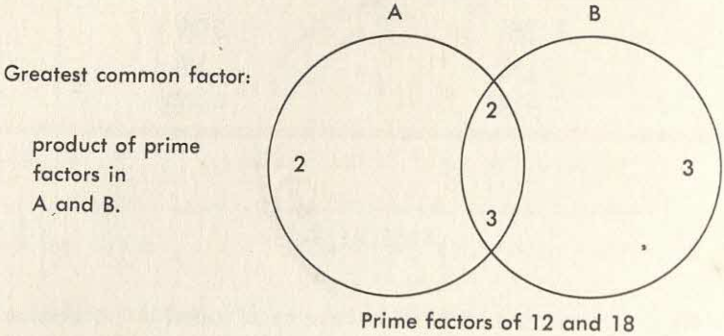


FIGURE 6–14

the prime factors of 36, Set B includes the prime factors of 48, Set C is the set of all whole number factors of 36, and Set D is the set of all whole number factors of 48.

Set A = {2, 2, 3, 3}   Set C = {1, 2, 3, 4, 6, 9, 12, 18, 36}

Set B = {2, 2, 2, 2, 3}   Set D = {1, 2, 3, 4, 6, 8, 12, 16, 24, 48}

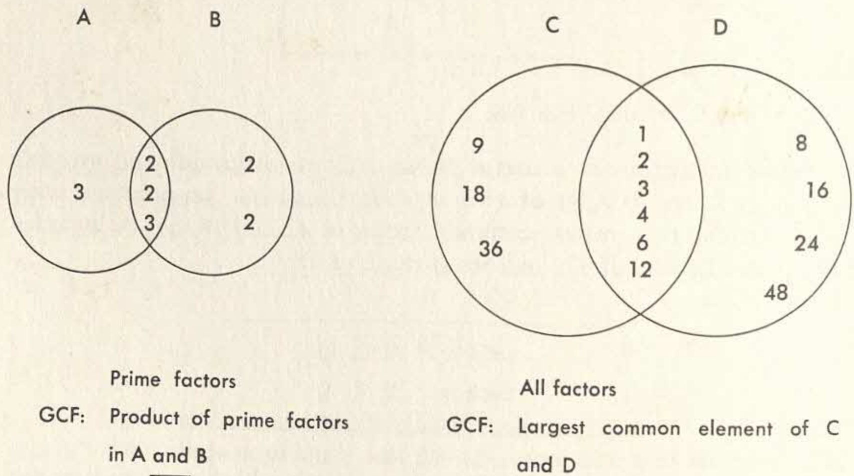


FIGURE 6–15

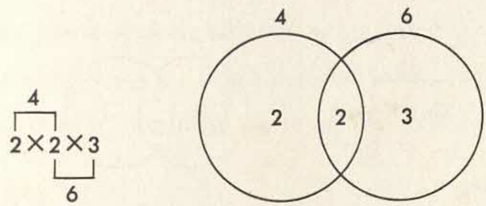
Least Common Multiple

A multiple of a whole number is the product of that number and some other whole number. Thus, the common multiple of two num-

bers can be determined by inspecting sets of multiples of each of the numbers. Multiples of 4 and 6 are shown in Sets A and B, respectively.

Set A =	{0, 4, 8, 12, 16, 20, 24, 28, 32, 36, . . .}
Set B =	{0, 6, 12, 18, 24, 30, 36, 42, . . .}

It can be readily seen that common multiples in the two sets (as far as multiples are actually shown) are 0, 12, 24, and 36. Excluding 0, the least of these common multiples is 12. The least common multiple of two or more non-zero whole numbers is taken to be the least non-zero whole number which is a multiple of each of the given numbers. Primes can be used to determine the least common multiple. Any multiple of 4 must contain the prime factors of 4 (2 and 2). Any multiple of 6 must contain the prime factors of 6 (2 and 3). The set of primes with the smallest number of elements which includes the set of prime factors for both numbers would be {2, 2, 3}.



The prime factor 2 is in the intersection of the sets of prime factors for the two numbers 4 and 6. The least common multiple for 4 and 6 is the product of the prime factors in the union of the two sets. The least common multiple for each of several pairs of numbers is shown in Figure 6–16 on the following page.

EXERCISES

1. Is the set of even numbers closed under multiplication?
2. Is the set of even numbers associative and commutative under multiplication?
3. Does the distributive property of multiplication with respect to addition hold for the set of even numbers?
4. Is the set of odd numbers closed, associative, and commutative under multiplication?

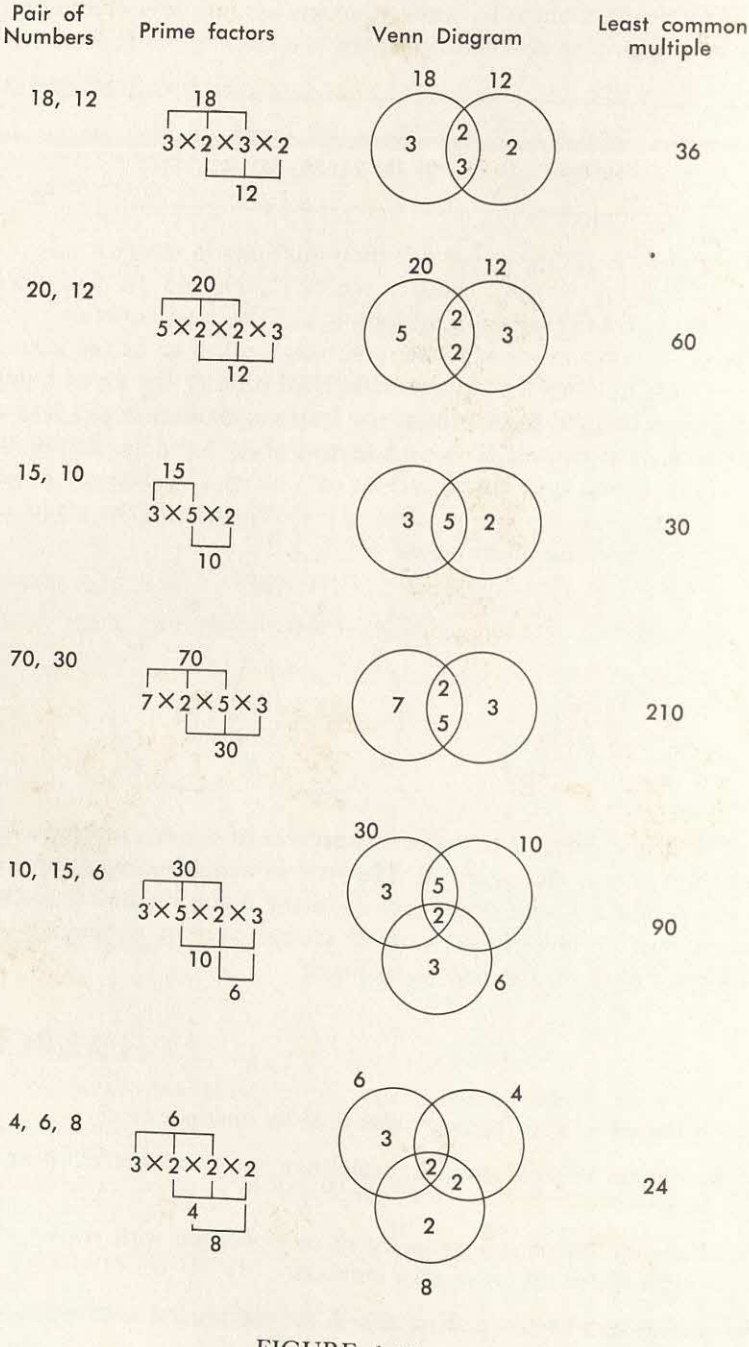

Pair of Numbers
Prime factors
Venn Diagram
Least common multiple
18, 12
18
3×2×3×2
12
18
12
3
2
3
2
36
20, 12
20
5×2×2×3
12
20
12
5
2
2
3
60
15, 10
15
3×5×2
10
3
5
2
30
70, 30
70
7×2×5×3
30
7
2
5
3
210
10, 15, 6
30
3×5×2×3
10
6
30
10
3
5
2
3
6
90
4, 6, 8
6
3×2×2×2
4
8
6
4
3
2
2
2
8
24

FIGURE 6–16

5. Does the distributive property of multiplication with respect to addition hold for the set of odd numbers?

6. Draw a triangular number pattern for the sum of the counting numbers to 10; to 12.

7. Draw a square number pattern which is the combination of the triangular number patterns representing the sum of the counting numbers to 5 and to 6.

8. Draw as many different arrays as you can to illustrate the number 6; the number 7; the number 10; the number 16; the number 23; and the number 24.

9. Are any of the triangular numbers also rectangular numbers?

10. Is there a single number less than 100 which can be shown as a triangular number, a square number, and a rectangular number other than a square number?

11. What are the first three composite numbers after 12?

12. What are the first three prime numbers after 12?

13. What are the prime numbers between 20 and 30?

14. Which of the following is a prime number: 21, 39, 42, 43, 49, 51?

15. Draw factor trees to determine the prime factors for each of the following numbers.

    a. 18
    b. 48
    c. 42
    d. 27
    e. 51
    f. 126
    g. 75
    h. 450
    i. 385
    j. 143

16. Use the successive division method to determine the prime factors for each of the following numbers.

    a. 72
    b. 16
    c. 35
    d. 48
    e. 60
    f. 128
    g. 561
    h. 243
    i. 256
    j. 2310

17. Determine the greatest common factor for each pair of numbers using sets of prime factors.

    a. 8, 12
    b. 10, 15
    c. 154, 245
    d. 54, 135

18. Determine the greatest common factor for each pair of numbers using Venn Diagrams.

    a. 25, 35
    b. 16, 25
    c. 96, 224
    d. 250, 375

19. Determine the greatest common factor for each pair of numbers using sets of all factors.

    a. 12, 36
    b. 45, 30
    c. 128, 144
    d. 175, 250

20. Determine the least common multiple for each pair of numbers using sets of prime factors.

    a. 6, 10
    b. 12, 30
    c. 20, 30
    d. 30, 45

21. Determine the least common multiple for each pair of numbers using Venn Diagrams.

    a. 9, 15
    b. 12, 28
    c. 24, 40
    d. 36, 54

22. Determine the least common multiple for each pair of numbers using sets of multiples of each.

    a. 12, 36
    b. 5, 7
    c. 6, 10
    d. 6, 14

23. Determine the greatest common factor and the least common multiple for each pair of numbers.

    a. 8, 16
    b. 12, 15
    c. 9, 24
    d. 14, 35

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Chapter 7

Concepts to be developed in this section are:

1. *Addition and multiplication of common fractions must be defined.*
2. *The meaning of equality of common fractions plays a central role in both the reduction and division of fractions.*
3. *The properties associated with the manipulation of fractions are man-made.*
4. *There are many ways in which the child experiences the fraction concept.*

7

Understanding Fractions

I. COMMON FRACTIONS

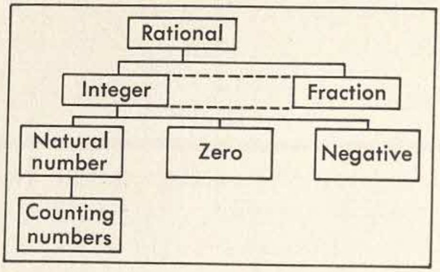
RELATING FRACTIONS TO THE NUMBER SYSTEM

What is a common fraction? This question may cause us to deliberate if we consider the following: Is the number three ever a common fraction? Is the number three ever equal to a common fraction? Is a common fraction ever negative? Is a "mixed number" a common fraction?

The integers are defined to be the natural numbers (counting numbers), their negatives, and zero. A rational number is one which can be expressed as the quotient of two integers (excluding zero as a divisor). Common fractions are rational numbers, and even the integers may be expressed as common fractions. For example, the number three can be expressed as

$$\frac{3}{1},\ \frac{6}{2},\ \frac{21}{7}\text{, etc.}$$

While this gives us a definition of common fractions, it leaves something to be desired in that it does not tell us much about how we



encounter fractions in the world in which we live. That is, what are the ways in which fractions may be interpreted? Various books break down the interpretations into many different parts.

The following are interpretations of the fraction $\frac{a}{b}$, taken from Schaaf[1] and illustrated by applications to concrete situations.

1. Dividing *a* elements into *b* equal sets, and asking how many elements there are in each of the equal sets

1. A group of 18 boys is to be divided into 3 equal sets. How many boys are in each set? $\frac{18}{3} = 6.$

2. Dividing *a* elements into equal sets, each set to have *b* elements, and asking how many such sets there will be

2. The 28 members of a class are to be divided into groups of 4 children. How many such groups are there? $\frac{28}{4} = 7.$

3. Dividing one whole into *b* equal parts and taking *a* of these parts

3. One pie is to be divided into 8 parts. Jack is to be given 3 pieces. What part of the pie will he have? He will have $\frac{3}{8}$ of a pie.

4. Dividing each of *a* wholes into *b* equal parts, and then taking one part from each whole

4. Each of 3 apples is to be divided into 4 equal parts, and then one part is to be taken from each of the 3 apples. How much apple will have been taken? $\frac{3}{4}$

[1] Schaaf, *Basic Concepts of Elementary Mathematics*, pp. 200-201.

5. Comparing *a* with *b* in order to give ratio of sizes	5. John has \$3; Tom has \$2. If they buy toy soldiers, John will have $\frac{3}{2}$ as many as Tom.

If we look now at the definition and interpretations of common fractions given above, we see the answers to some of the questions with which we opened the chapter. In the light of the definition, for example, we see that the number three may be thought of as a fraction. It may be that to some it would be preferable to represent three as $\frac{3}{1}$ to see that it can be made to look like a fraction numeral. Common fractions can be negative, since some of the rational numbers are negative.

There are several ways in which numerals which represent fractions may be written (see Figure 7–1). One way a common fraction may be represented is as a numerator and a denominator separated by a horizontal line. This is probably the best way to write it when we are teaching children. The notation ⅗ probably should not be introduced until the child is very familiar with and proficient in the use of the notation $\frac{3}{5}$.

SOME NOTATIONS FOR FRACTIONS

$\frac{3}{5}$ 3/5 $3 \div 5$

$\frac{\text{numerator}}{\text{denominator}}$

FIGURE 7–1

A third notation is in use in modern mathematics to denote the fraction $\frac{a}{b}$. While the notation (a, b) lends itself to a more rigorous treatment of the subject of rational numbers, it has many disadvantages. It is easily confused with the same notation which is used for many other purposes. While it is important to recognize that the notation is man-made, it is most desirable that we not stray too far from the conventional when we teach children unless there is an overwhelming

advantage to abandoning the conventional. Hereafter, we will use the conventional fraction notation (that is, $\frac{a}{b}$).

EQUALITY OF FRACTIONS

Let us now consider the subject of the equality of two fractions. By definition $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. This is the definition of the equality of two fractions even if numerator and/or denominator are themselves fractions. Since *division* is defined in terms of multiplication, and since fractions denote division, it follows logically that the equality of fractions should be defined in terms of multiplication.

Two consequences result from this definition of the equality of two fractions. The first of these is that it is unnecessary to reduce two fractions to a common denominator to determine their equality. As an example, are the fractions $\frac{4891}{3131}$ and $\frac{73}{101}$ equal? It would be possible, and not even very difficult, to obtain a common denominator, and then see if the numerators are equal, but it is much more direct to turn the question into: "Is $4891 \times 101 = 3131 \times 73$?"

The second consequence of the definition is possibly more useful. In fact, we use it almost every time we work with fractions. It is the ability to reduce fractions. We can state the property in general as: $\frac{ac}{bc} = \frac{a}{b}$. That is, if numerator and denominator possess a common factor, then both may be divided by this factor without altering the value of the fraction. The fact that the two fractions are equal follows from the definition of equal fractions and the commutative property of multiplication. That is, $acb = abc$. We sometimes speak of cancelling the common factor and indicate it as $\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b}$. The term "cancel" is not entirely improper *if* we recognize its significance, but it is probably more meaningful to children to speak of dividing both numerator and denominator by a common factor.

ADDING AND SUBTRACTING FRACTIONS

With this background, let us turn our attention to the addition of fractions. The sum of two common fractions is defined: $\frac{a}{b} + \frac{c}{d} =$

$\frac{ad+bc}{bd}$. This definition is not a direct consequence of the definition of equality of fractions, since before the definition of sum was given, it was not known that the sum of two fractions was a fraction. A teacher who desires to investigate the consistency of this definition and properties of arithmetic may find further discussion in many books—for example, Schaaf, Chapter 4.

We have made modifications of this definition using the definition of equal fractions. An example will serve to illustrate this point. Let us add $\frac{2}{15}+\frac{7}{9}$. By definition, the sum is $\frac{2}{15}+\frac{7}{9}=\frac{2\cdot 9+7\cdot 15}{9\cdot 15}=\frac{18+105}{135}=\frac{123}{135}$. However, this answer may be reduced, by applying the second consequence of the definition of equal fractions, since both numerator and denominator contain a factor of 3.

$$\boxed{\frac{123}{135}=\frac{41\times 3}{45\times 3}=\frac{41}{45}}$$

We have learned that it is not necessary to find the sum before reducing but that the reduction may be made initially by writing each fraction with a *least common denominator*. In our example the least common denominator (called, in algebra, the least common multiple of the denominators) may be found as follows: $\frac{2}{15}+\frac{7}{9}=?$ We write $9=3\cdot 3$ and $15=3\cdot 5$. Now, the least number into which both 9 and 15 may be divided an integral number of times is $3\cdot 3\cdot 5$ – the $3\cdot 5$ accommodating the division by 15, and the $3\cdot 3$ that by 9. We now know the *least* common denominator is 45. Our next step is to express fractions $\frac{2}{15}$ and $\frac{7}{9}$ each with a denominator of 45. To do this we reverse the steps for reducing fractions and multiply both numerator and denominator by appropriate equal factors (still another consequence of the definition of equality of fractions is that if $\frac{ac}{bc}=\frac{a}{b}$, then $\frac{a}{b}=\frac{ac}{bc}$). Thus, $\frac{2}{15}=\frac{2\cdot 3}{15\cdot 3}=\frac{6}{45}$ and $\frac{7}{9}=\frac{7\cdot 5}{9\cdot 5}=\frac{35}{45}$. Therefore, $\frac{2}{15}+\frac{7}{9}=\frac{2\cdot 3}{15\cdot 3}+\frac{7\cdot 5}{9\cdot 5}=\frac{6}{45}+\frac{35}{45}$. Now, how are we justified in adding these fractions, which have common denominators, by adding numera-

tors only? By definition, it would seem that $\frac{a}{b} + \frac{c}{b} = \frac{ab + cb}{b \cdot b}$. How is it that this is the same as $\frac{a + c}{b}$? Is $\frac{ab + cb}{b \cdot b} = \frac{a + c}{b}$? The answer is immediate, and it may be seen in this way: we have $\frac{ab + cb}{b \cdot b} = \frac{b(a + c)}{(b \cdot b)}$ by use of the distributive property, therefore, $\frac{b(a + b)}{(b \cdot b)} = \frac{a + c}{b}$. This last step is accomplished by dividing numerator and denominator by the common factor b (a valid operation, as seen earlier, by the definition of the equality of fractions). Since these quantities are equal, we find that $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$ as a consequence of our properties and definitions. We know now that we may go back to $\frac{6}{45} + \frac{35}{45}$ and add numerators only, retaining the common denominator, to obtain $\frac{41}{45}$. We all do our addition of fractions by obtaining least common denominators when feasible, but how often have we thought of why this is a valid operation?

An example using pie diagrams may serve to clarify the addition of fractions. Let us take the problem $\frac{1}{3} + \frac{1}{4}$, illustrated in Figures 7–2 and 7–3. We can put the two pieces of pie together to obtain a new part of pie, but unless we are able to find some common unit of measure we are hard pressed to decide what fractional part of a pie is obtained.

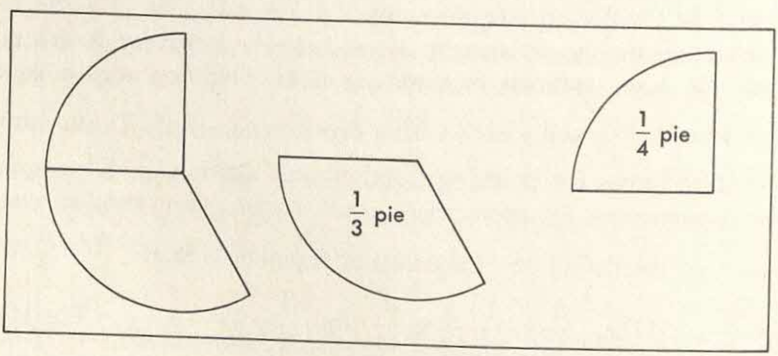
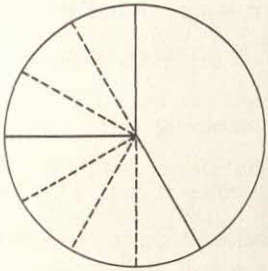


FIGURE 7–2

Brief thought will show that each of the pieces ($\frac{1}{3}$ and $\frac{1}{4}$) may be subdivided into parts of equal size if the $\frac{1}{3}$ pie is cut into 4 twelfths and the $\frac{1}{4}$ pie into 3 twelfths. It is easy to visualize this process with reference to our example. A few other examples should serve to show that the procedure is independent of the type of material being added, but is rather a procedure for adding the fractions themselves.



$$\frac{1}{3}+\frac{1}{4}=\frac{4}{12}+\frac{3}{12}=\frac{7}{12}$$

FIGURE 7–3

Before proceeding to the other fundamental operations, let us look at a way by which children sometimes try to add fractions. We need to know why we must reject this method if we are to be better able to counsel our pupils about not using it. Suppose we were to define the sum of two fractions as: $\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$. First, we see that we would want to reject this definition because it gives us answers which are inconsistent with the way we want our number system to describe the physical world in which we live. Let us consider adding the fractions $\frac{2}{3}$ and $\frac{1}{3}$ by this definition. This would give $\frac{2}{3}+\frac{1}{3}=\frac{3}{6}$. Now, we do not know that $\frac{3}{6}$ in this system is equivalent to $\frac{1}{2}$ but we surely would want $\frac{3}{6}$ to be the same as $\frac{1}{2}$. We find that $\frac{2}{3}+\frac{1}{3}$ by this definition does not give a reasonable answer, so we would want to reject this definition.

Before casting this definition aside completely, let us look at a rea-

sonable use for it. The teacher meets this peculiar type of addition occasionally. Her students do, too. An example of the situation in which the students encounter it will serve as an introduction and a brief illustration will follow to show how the teacher also encounters this type of problem.

In Schaaf, an example like this is given. Yesterday a baseball player went to bat 5 times and got 3 hits. This may be interpreted as saying he got a hit $\frac{3}{5}$ of his times at bat. Today he batted 7 times and hit 4. This may be represented as $\frac{4}{7}$. Now, in finding his batting average for the two days, we certainly would not want to add these fractions in the usual way: $\frac{3}{5}+\frac{4}{7}=\frac{21+20}{35}=\frac{41}{35}$ – or, more than a 1000 batting average. What we really mean, in combining these fractions, is $\frac{3}{5}+\frac{4}{7}=\frac{7}{12}$. That is, he hit 7 times in 12 times at bat. This procedure in addition yields the correct result because of the fact that, in the sum, we want the *whole* to be his 12 times at bat, and the figures are really revised to mean 3 hits and 4 hits in 12 times at bat = 7 hits in 12 times at bat.[2]

The teacher also encounters this type of addition. Suppose she administered a two-part test to her class on two separate days. John scored 42 correct out of 50 on the first day and 31 correct out of 40 on the second day. The meaningful interpretation of these data in terms of fractions would be that $\frac{42}{50}+\frac{31}{40}=\frac{42+31}{50+40}=\frac{73}{90}$. That this score differs from the one which would have been obtained by the ordinary averaging method may be verified by anyone who is interested. Again we have written one thing $\left(\frac{42}{50}\right)$ when we meant another $\left(\frac{42}{90}\right)$.

Subtraction of fractions may be passed over with very little comment since the definition (or rule) for subtraction conforms exactly to that for addition. That is, $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$.

An example is: $\frac{5}{7}-\frac{4}{9}=\frac{5\cdot 9-4\cdot 7}{63}=\frac{45-28}{63}=\frac{17}{63}$.

[2] Schaaf, *Basic Concepts of Elementary Mathematics*, pp. 201-202.

MULTIPLYING AND DIVIDING FRACTIONS

The operation of multiplication of fractions is governed by the following definition: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. This definition, like those for addition and subtraction before it, conforms to the ordinary meaning of multiplication of integers in the event b and d are each equal to 1. If we interpret the times sign to mean "of," it also conforms to our observed behavior of the world around us. For example, $\frac{1}{2}$ of $\frac{1}{3}$ of a pie is $\frac{1}{6}$ of a pie.

Using this definition, and that of the equality of fractions, we find that for every fraction $\frac{a}{b}$ (except when $a = 0$ – we have already excluded $b = 0$) there exists another fraction $\frac{b}{a}$, called the *inverse* of $\frac{a}{b}$, for which $\frac{a}{b} \cdot \frac{b}{a} = 1$. The multiplicative inverse is also frequently called the reciprocal. This inverse, or reciprocal, has great significance when we get to the subject of division.

It is important, after having defined multiplication of fractions, to help the child relate this definition to the world around him. In this relation it is desirable to show how the commutative principle is applied. As an example, consider $3 \cdot \frac{1}{2} = \frac{1}{2} \cdot 3$. The first part of this equality might be illustrated to mean "take three half-apples" and the other to mean "take half of three apples." It is not hard to show the child that these real objective quantities are equal. A pie chart might also be used to show that $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ or that $\frac{2}{3} \cdot \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$.

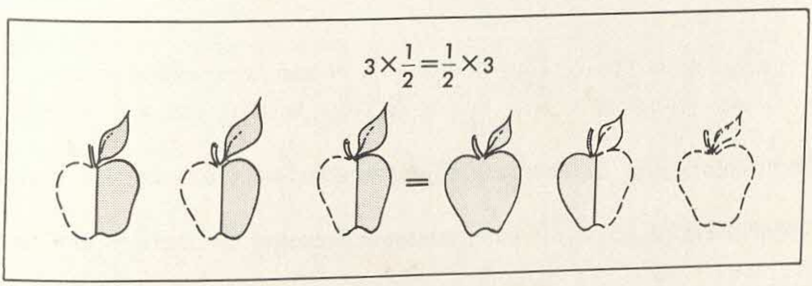


FIGURE 7–4

Another application of the multiplication of fractions may be illustrated with areas. This illustration may not be well understood until the class has had some experience with areas. Let us multiply $\frac{2}{3} \cdot \frac{3}{7}$. First, we take a rectangle 7 units long and 3 units wide. We divide it into 21 squares as in Figure 7–5. The product of $\frac{2}{3}$ and $\frac{3}{7}$ may be associated with the areas of the six squares which are shaded. These six squares constitute $\frac{6}{21}$ of the entire area. That is, $\frac{2}{3} \cdot \frac{3}{7} = \frac{6}{21}$.

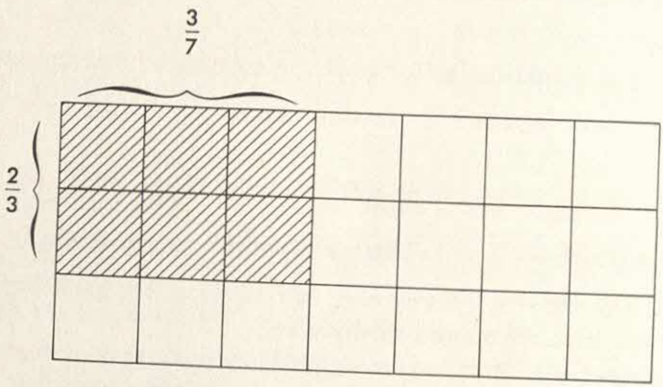


FIGURE 7–5

The rule for dividing fractions is: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$. We have formalized its statement by saying that we invert the divisor and multiply. Too often when the question, "Why?" is asked, the answer has been, "Because the rule says so." Is this rule a definition, or is it a consequence of previous definitions? It is a consequence. Let us consider the problem written as $\frac{N}{D} = \frac{\frac{a}{b}}{\frac{c}{d}}$ in which N and D represent numerator and denominator, respectively. The fraction $\frac{c}{d}$ has a multiplicative inverse or reciprocal $\frac{d}{c}$. By the meaning of equality of fractions, we may multiply both numerator and denominator by $\frac{d}{c}$ obtaining:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}} = \frac{\frac{ad}{bc}}{1} = \frac{ad}{bc}$$

We can now say that in dividing fractions, inverting the divisor and multiplying is a good description of the rule, but that it is a consequence of previous rules of operation.

This rule may be illustrated by the use of the pie chart in Figure 7–6. We can ask, "How many $\frac{2}{5}$ are there in $\frac{4}{5}$?" First take $\frac{4}{5}$ of the pie chart (B). How many $\frac{2}{5}$ are in $\frac{4}{5}$? Diagram C shows there are two $\frac{2}{5}$ in $\frac{4}{5}$. Division by rule also gives $\frac{4}{5} \div \frac{2}{5} = \frac{4}{5} \cdot \frac{5}{2} = \frac{4 \cdot 5}{2 \cdot 5} = \frac{2 \cdot 2 \cdot 5}{2 \cdot 5}$ $= 2$. Through many other illustrations of this type the pupil can relate the rule for dividing fractions to observed facts about his environment.

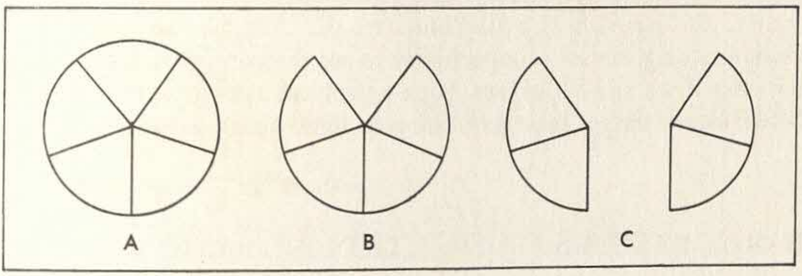


FIGURE 7–6

Another illustration which can be explained to children quite easily is this: How many $\frac{3}{4}$ are in 5? This is a very good way to read the problem $5 \div \frac{3}{4}$. We ask first, how many $\frac{3}{4}$ are there in 1? Using a pie diagram, we can see that more than one $\frac{3}{4}$ of a pie is contained in 1 pie. In fact, 1 pie is made up of one $\frac{3}{4}$ pie and one $\frac{1}{4}$ pie. This $\frac{1}{4}$ pie is $\frac{1}{3}$ of a $\frac{3}{4}$ pie, so 1 pie contains one and one-third $\frac{3}{4}$ pie. One and one-third is four-thirds. So, 1 pie contains four-thirds $\frac{3}{4}$ pies. We see,

then, that 5 pies would contain 5 times as many $\frac{3}{4}$ pies. That is, $\frac{5}{\frac{3}{4}} = 5 \cdot \frac{4}{3}$. In this example we see the divisor inverted and multiplied—the common procedure for dividing by a fraction.

In more complex situations, for example, $\frac{8}{11} \div \frac{7}{9}$, we can obtain equivalent fractions containing common denominators. The problem can then be explained by one of the above methods. Care should be taken, though, not to leave the impression that obtaining a common denominator is a necessary step in the division of fractions.

A word of warning is in order at this point. In teaching elementary school children, we must relate their learning experience with numbers to their experiential world, but we should also emphasize that the rules of mathematics are man-made and, consequently, that any relation between mathematics and the physical world is a relation which is man-made. We do quite generally desire that our mathematical laws conform to the experiential world, but after the child has learned the laws, he should be given some opportunity to see the consequences of assuming other laws and definitions. Such a problem as that given earlier for computing averages may serve to give him this added experience.

EARLY SYSTEMS OF FRACTION NOTATION

For many it may be of interest to conclude with some remarks about the way fractions were treated by the ancients. We might note first that "fraction" means "broken number." We are familiar with the related word fracture. The ancient Egyptians wrote all fractions with unit numerators. Any fraction which could not be expressed directly with a unit numerator was written as the sum of unit fractions. As an example, using our notation $\frac{7}{24}$ would have been written as $\frac{1}{4} + \frac{1}{24}$, and $\frac{23}{24}$ would have been $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{12}$ (or some other combination of unit fractions). Even today we do much the same sort of thing when we make change for amounts less than \$1.00. For example, $73¢ = \frac{73}{100}$ dollars $= \frac{1}{2} + \frac{1}{10} + \frac{1}{10} + \frac{1}{100} + \frac{1}{100} + \frac{1}{100}$ dollars, all fractions being usable unit parts of a dollar.

The Romans had fractions which for the most part had denominators of twelve. Their symbols for these fractions were not as clumsy as those of the Egyptians, but they were unwieldy by our standards. The Hindus developed a system of notation for fractions much like that which we use today except for one detail. They did not use a horizontal line to separate numerator and denominator. They used horizontal positioning of digits to indicate whole numbers and vertical positioning to indicate fractions, much as we use today. The very ancient Greeks had no notation for fractions because of a philosophical drawback. Brilliant as many ancient Greek mathematicians were, they associated unity with deity. They could not reconcile themselves to a philosophy which would allow unity to be divided into parts, for this was an insult to their gods as well as being impossible in their minds. This atomistic philosophy–in which there is no part smaller than an atom and no fraction of unity–gradually gave way to the pressure exerted by the more practical Greek merchants, who had no qualms about using fractions in trade. By the time of Archimedes, ratios were in common use in Greek mathematics.

II. DECIMAL FRACTIONS

Concepts to be developed in this section are:

1. *The units place, not the decimal point, is the pivotal position.*
2. *Estimating decimal position in multiplication and division is important in early decimal work.*
3. *Decimal notation—past and present—has been represented in a variety of ways.*
4. *Differences between common fractions and decimal fractions are differences in notation only.*
5. *Decimal fractions may be expressed as common fractions. Some common fractions may be expressed as decimal fractions, and all may be approximated by decimal fractions.*

Decimal fractions are used regularly by all of us. From odometers to our paychecks, decimal fractions are serving man.

Although the Babylonians used some elements of decimals in their numeral system, it was not until 1585 that the decimal idea was introduced with our modern base ten numeral system by the Belgian, Simon Stevin. Stevin used a numbering system to indicate place value in decimal notation. With this system, we would indicate the decimal notation for 68.345, for example, in the following way.

STEVIN SYSTEM

⓪	①	②	③
6 8	3	4	5

DECIMAL USES

Odometer	1 2 4 6 8 3
Clinical thermometer	98.6°
Batting average	.275
Micrometer reading	.0014″
Longitude	108° 14′ 36.25″
Average weight	107.3 lbs.
Grade point average	2.84
Percentages	16.5%
Dollars	$12.69

FIGURE 7–7

Many methods of indicating the placement of the units column in decimal expressions have been used. At various times the decimal numeral 68.345 has been written in the ways shown below.

Even today the method of notating decimal fractions varies from place to place. The primary difference is in the way by which the units

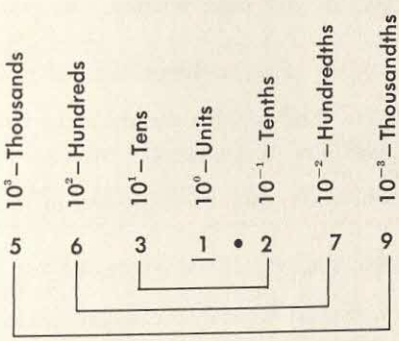
6 8 ⓪ 3 ① 4 ② 5 ③ $\quad 68\,\underline{345} \quad 68\underline{/345}$

$\overset{o}{68}\,.\,\overset{i}{3}\,.\,\overset{ii}{4}\,.\,\overset{iii}{5}$

$68/\dot{3}\dot{4}\dot{5} \quad 68\overline{\smash{)}\,345}^{\,123} \quad 68,\ 3'4''\ 5'''$

digit of the numeral is identified. In the United States the mark which identifies the units digit is a period following the units digit. This notation, like the others, has drawbacks. We can see how it might cause confusion to end a sentence with the numeral 3.14, for example, followed by a period used as a punctuation mark. In Great Britain the same type of mark is used, but it is elevated somewhat above the position a period would occupy; sometimes it is printed with a slightly larger dot than a period—for example, 3 · 14. This, too, is awkward in that the decimal point may be mistaken for a times sign. In Germany and France and in many of the countries of the Near and Far East, a comma is used instead of a period. Thus, our 3.14 would appear as 3,14. This would be awkward if the numeral were 3,142 for it might be read as three thousand one hundred forty-two rather than three and one hundred forty-two thousandths.

Before proceeding further with a study of the decimal system of notation, it should be made very plain that the difference between common fractions and decimal fractions is purely a difference in notation. In this section we are focusing on the problems which are raised by this decimal notation.



The identifying mark by which we determine the units digit is called the decimal point (in a quinary system of numeration it would be the quinary point) or the separatrix. With the decimal point to show the separation of whole number from fraction we are able to diagram the place values associated with the digits. The diagram clarifies that it is the units digit which is pivotal and *not* the decimal point. On a number scale the zero position is usually the pivot point. In the scale denoted by the exponents and shown above the positions of the digits of the numeral in the above diagram, the units digit is associated with the zero exponent (applied, of course, to the base 10).

There is no fixed rule for reading decimal numerals orally; a com-

mon method is to insert the word "and" to express the decimal point, and then not use "and" anywhere else in the reading of the numeral. As an example, 324.162 would be read "three hundred twenty-four and one hundred sixty-two thousandths." This is also frequently—and for all useful purposes, correctly—read as "three two four point one six two." This latter method is not suggested for use in the elementary school for it gives the reader and listener no practice in the use of the correct place value names of the digits, and if used consistently might ultimately lead to confusion, since little understanding of place value associations would be emphasized.

In some textbooks, mixed decimal and common fractions are used and advocated for use in the elementary grades. There is probably no mathematical foundation for criticism of such a method, but it does lead to certain misunderstandings and to the need for a larger set of rules to accomplish the four fundamental operations. For example, $.37\frac{1}{2}$ means the same as .375, and it would probably be better to write it in the latter fashion. When a common fraction is used to terminate a decimal fraction, it is considered to represent this fractional part of the place value of the last decimal digit written. For example, $.42\frac{1}{3}$ implies $.42 + \frac{1}{3}\ (.01)$ because the place value of the last decimal digit written is hundredths. This illustration alone should serve to explain why decimal and common fractions are frequently mixed. As we will soon see, the common fraction $\frac{1}{3}$ may not be expressed in decimal form with a finite number of decimal digits. If we wrote $.42\frac{1}{3}$ as .4233, for example, the latter expression would be only an approximation of the former.

In this connection, one branch of the study of decimal fractions in which mixtures of decimal and common fractions are frequently found is that of percentage. We are all familiar with the use of the word "per cent" or the symbol "%" to denote "hundredths." Thus, 16% means .16, and, conversely, .387 may be written as 38.7%. The rules which are used in positioning the decimal point are based on the meaning of percentage. In percentage notation, it is considered quite proper to use $16\frac{1}{4}\%$, actually a mixture of decimal and common fractions, since $16\frac{1}{4}\%$ means $.16\frac{1}{4}$ or .1625.

DECIMAL FRACTION ↔ COMMON FRACTION

These last considerations lead us to the topic of converting from decimal fractions to common fractions, and vice versa. Since every finite decimal expression is actually a common fraction whose denominator is some power of ten (meaning ten multiplied by itself a few or many times), conversion from decimal form to common fraction form is quite easy. The examples in Figure 7–8 illustrate how this conversion is accomplished. From these examples we see in several ways that the place value of the last decimal digit determines the denominator of the common fraction representing the same number represented by the decimal fraction.

A $.375 =$ three hundred seventy-five thousandths

$$= \frac{375}{1000}.$$

$$\text{Also, } .375 = \frac{3}{10} + \frac{7}{100} + \frac{5}{1000}$$

$$= \frac{300}{1000} + \frac{70}{1000} + \frac{5}{1000}$$

$$\frac{375}{1000}.$$

B $.00297 =$ two hundred ninety-seven hundred thousandths

$$= \frac{297}{100{,}000}.$$

C $$.42\tfrac{1}{3} = \frac{42}{100} + \frac{1}{3}\left(\frac{1}{100}\right)$$

$$= \frac{126}{300} + \frac{1}{300}$$

$$= \frac{127}{300}.$$

D $$321.75 = 321\frac{75}{100}$$

$$= \frac{32100}{100} + \frac{75}{100}$$

$$= \frac{32175}{100}.$$

FIGURE 7–8

Conversion of mixed numbers (in which the fractional part is expressed decimally) is accomplished in a similar way, and the answers may be expressed as mixed whole numbers and common fractions, or as improper fractions. Example D in Figure 7–8 should be sufficient illustration of this technique. The form of the last fraction in the illustration indicates that this improper common fraction could have been obtained by copying the entire numeral, digit for digit, and placing this numeral over the denominator indicated by the place value of the

final digit. We say "indicated by" since the place value of the final digit is hundredths while the *denominator* used is hundred.

In some of the examples given above, certain simplifications or reductions are in order. For example, in the expression $.375 = \frac{375}{1000}$, the numerator 375 and the denominator 1000 contain a common factor of 125, so the reduction of the common fraction form may be accomplished as $\frac{375}{1000} = \frac{3(125)}{8(125)} = \frac{3}{8}$. We sometimes say that we have "cancelled" the 125's, but it is more proper to say that we have divided both numerator and denominator by a common factor.

Conversion from common fractions to decimal fractions can be accomplished most readily where the multiplier necessary to convert the denominator of the given common fraction to an exact power of ten is immediately obvious. In this case, we can reverse the steps given above and multiply both the numerator and denominator by this multiplier. For example, in converting $\frac{3}{8}$ to decimal form, we may notice that multiplication of 8 by 125 gives 1000, which is an exact power of ten. We then proceed to multiply both the 3 and the 8 by 125. We obtain $\frac{3}{8} = \frac{3(125)}{8(125)} = \frac{375}{1000} = .375$, which is a decimal representation of the fraction. Other decimal representations, such as .3750 and .37500, are also possible. We frequently choose to cast aside terminal zeros, and merely write .375 for the result.

CONVERSION BY INSPECTION

$\frac{7}{10} = .7$

$\frac{7}{25} = \frac{7(4)}{25(4)} = \frac{28}{100} = .28$

$\frac{3}{5} = \frac{3(2)}{5(2)} = \frac{6}{10} = .6$

$\frac{33}{200} = \frac{33(5)}{200(5)} = \frac{165}{1000} = .165$

In the event that the multiplier necessary to convert the denominator of the given common fraction to an exact power of ten is not immediately obvious, another familiar procedure is available. This is to convert the numerator to a mixed decimal expression by putting in a decimal point followed by a number of zeros. Division of this mixed decimal numerator by the denominator then converts the fraction to

its decimal form. In doing this division it is necessary to assume a knowledge of some rule for determining the position of the decimal point in the result. This rule will be discussed presently, but for the time being the following device is sufficient.

$$\frac{3}{8} = \frac{3.0000}{8} = \frac{3.0000 \times 10{,}000}{8 \times 10{,}000} = \frac{30{,}000}{8}\left(\frac{1}{10{,}000}\right)$$

$$= 3750\left(\frac{1}{10{,}000}\right) = .3750 = .375.$$

The introduction of more than enough zeros in the second step was deliberate, for at the outset of most problems it is not known how many zeros may be needed. The denominator 10,000 was reserved in the fourth and fifth steps to position the decimal point in the result.

A second technique illustrated in Figure 7–9 allows for the introduction of extra zeros as needed. To convert $\frac{21}{56}$ to its corresponding decimal expression, we write the 21 as the dividend and the 56 as the divisor in an ordinary division algorithm. We then place a decimal point after the 21 and append zeros as needed as in the steps shown in the illustration. The decimal point in the quotient is located directly above the decimal point of the dividend since the divisor is a natural number.

```
     00
56 ) 21.

     00 3
56 ) 21.0
     16 8
        4 2

     00 3
56 ) 21.00
     16 8
      4 20

     00 37
56 ) 21.00
     16 8
      4 20
      3 92
        28

     00 37
56 ) 21.000
     16 8
      4 20
      3 92
       280

     00 375
56 ) 21.000
     16 8
      4 20
      3 92
       280
       280

     00.375
56 ) 21.000
```

Therefore, $\frac{21}{56} = .375.$

FIGURE 7–9

If the division never terminates with a zero remainder, or if it appears that it is not going to terminate, we frequently stop the division process at some convenient place and "round" the answer. Rounding will be discussed in a later section. If the quotient is not exact, though, the decimal representation is only approximately equal to the common fraction being represented.

THE FOUR FUNDAMENTAL OPERATIONS AND DECIMAL FRACTIONS

$$\begin{array}{r} 34.3 \\ +7.89 \\ \hline 42.19 \end{array}$$

Let us now consider the four fundamental operations as applied to numbers represented in decimal notation. In adding and subtracting numbers represented in decimal notation, the rule states that we align the decimal points. Why? Two good answers may be given to this question. They both have their foundation in the same fact. The digits of the decimal representation all have place value. When we align the decimal points, we are simultaneously aligning all digits with like place value, and, in particular, we are aligning the pivotal units digits. In the accompanying problem, we see that the tens digits of the addends are aligned (since 7.89 has no tens digit), the units digits 4 and 7 are aligned, the tenths digits 3 and 8 are aligned, and the hundredths digits are aligned (the hundredths digit of 34.3 is understood to be 0 even though it is not expressed. "Carrying"—even across the decimal point—means just what it did before. It means "exchanging" ten of one type unit for one unit of the next greater place value. That is, in adding .3 and .8 we obtain eleven tenths, so we exchange 10 tenths for 1 unit or 1 one even though this means crossing over the decimal point. By this scheme it is easy to see that the decimal point in the sum is aligned with the decimal point in the addends.

$$\begin{array}{r} 1.739 \\ -.427 \\ \hline 1.312 \end{array} \qquad \begin{array}{r} 1 \\ 1.591 \\ -.427 \\ \hline 1.164 \end{array}$$

The rule for subtraction follows directly from the rule for addition since subtraction is defined in terms of addition. That is, in subtracting .427 from 1.739, we align the decimal points and subtract. We align the decimal point in the answer (the difference) with those in the subtrahend and minuend. If in another problem it is necessary to borrow, we are merely exchanging one unit of higher place value for ten units of lower place value. In subtracting .427 from 1.591 it is necessary to exchange one hundredth for ten thousandths in order to create eleven thousandths from which to subtract seven thousandths.

In the accompanying diagram the small 1 shown as borrowed means exactly this.

The second good reason which can be given for aligning the decimal points in addition and subtraction is that the process of aligning is equivalent to obtaining a common denominator in common fractions. For example, $7.21 + 3.427$, when written with decimals aligned, implies $7\frac{210}{1000} + 3\frac{427}{1000}$ which gives $10\frac{637}{1000}$ or 10.637. In the addition of 34.3 and 7.89 we had an example of what are called "ragged" decimals. Much can be said about this point, and in another section, comments will be made about accuracy of measured quantities and the expression of this accuracy by use of decimal notation. In many situations, we would say that the 7.89 should be rounded to show no more places of accuracy than the one place exhibited by the 34.3 before addition is performed. At this point it is sufficient to say that instances do occur in which "ragged" decimals may be added.

$$\begin{array}{rl}
1/2 = & .5 \\
1/4 = & .25 \\
3/8 = & .375 \\
1/3 = & .33333\ldots \\
\hline
 & 1.45833\ldots
\end{array}$$

When we ask a child to add $\frac{1}{2}, \frac{1}{4}, \frac{3}{8}$, and $\frac{1}{3}$, no thought is given to the accuracy of the measurements of quantities. A rule is developed by which such fractions may be added. Similarly, we have discussed a rule by which such "ragged" decimals may be added. The same rule is equally applicable even when the decimals are not "ragged."

The rule for locating the decimal point in the product when numbers given in decimal representation are multiplied also goes back to the meaning of decimal representation. Children's understandings are enhanced when they are helped to see this relationship between decimal and common fractions. For example, $.37 \times .4$ means $\frac{37}{100} \times \frac{4}{10}$. When these fractions are multiplied, the numerators are multiplied and the denominators are multiplied giving $\frac{148}{1000}$. The decimal representation of this product is .148, and it can be seen that the place value of the final digit is thousandths (the third decimal digit place value). The denomi-

nator 1000 is a consequence of multiplying 100×10, and the number of decimal places so determined for the decimal representation of the product is seen to be the sum of the numbers of decimal places in the multiplier and multiplicand. (That this rule is perfectly general is easily seen from the laws of exponents. The denominator of the common fraction form of the multiplier will be of the form 10^m, in which m is a natural number determined by the particular fraction. Similarly, the denominator of the common fraction form of the multiplicand may be expressed as 10^n for some natural number n. When multiplied, $10^m \times 10^n$ gives 10^{m+n}. That is, the denominator of the product involves 10 with an exponent which is the *sum* of the previous exponents, and this *sum* determines the number of decimal places in the product of the numbers.)

Problem	Approximate problem by	Answer should be near	Answer should not be near
$2.46\,\overline{)\,67.3}$	$\frac{67}{2}$	30	3 or 300
$27.9\,\overline{)\,547.8}$	$\frac{500}{25}$	20	2 or 200
$597\,\overline{)\,7.84}$	$\frac{8}{600}$	$\frac{1}{100}$	$\frac{1}{1000}$ or $\frac{1}{10}$

A similar rule for determining the position of the decimal point in a quotient may be stated, but the exceptions which must be stated with the rule make it unwieldy, so an alternative technique will be discussed here. First, let us take an example. We divide 37.68 by 24. Before concerning ourselves with developing the exact rule for determining the position of the decimal in the quotient, let us take a broad look at the problem. The position of the decimal, whatever it may be, affects the answer by powers of 10. That is, 3.14 is only one-tenth of 31.4, and 314 (with the decimal understood to be to the right of the digit 4) is one-hundred times as large as 3.14. Consequently, an estimate of the size of a quotient should give us the ability to position the decimal in

the quotient for the problem $37.68 \div 24$. We know that 24 will divide 37 only one (whole) time, so that the decimal in the quotient should follow the first non-zero digit. A little practice at such estimation should help the child see better why the rule to be developed here is valid.

To develop an exact rule for positioning the decimal point in the quotient, we first divorce the positioning problem from the division problem, and divide 3768 by 24. The quotient is 157. Now, returning to the problem involving the decimal point, we have: $\frac{37.68}{24}$ means

$$\frac{3768}{100} \div 24 = \frac{3768}{100} \times \frac{1}{24} = \frac{3768}{24} \times \frac{1}{100} = 157 \times \frac{1}{100} = \frac{157}{100} = 1.57.$$

$$\begin{array}{r} 1.57 \\ 24\,\overline{)\,37.68} \\ \underline{24} \\ 13\,6 \\ \underline{12\,0} \\ 1\,68 \\ \underline{1\,68} \end{array}$$

In this example, in which the divisor is a natural number, the number of decimal places in the quotient is the same as the number of decimal places in the dividend. This statement is true in general when the divisor is a natural number. Let us observe a way by which this fact may be illustrated in a particular form of the division algorithm. The decimal in the quotient is aligned with the decimal in the dividend (when the divisor is a natural number) if the left-hand digit of the quotient is placed directly above the right-hand digit of the first subtraction performed in the division algorithm.

$$\frac{3.768}{.24} = \frac{3.768 \times 100}{.24 \times 100} = \frac{376.8}{24}$$

This observed fact enables us to align the decimals in the division algorithm for any two numbers stated in decimal form. Let us justify the technique first. The problem $3.768 \div .24$ can be written as $\frac{3.768}{.24}$. Now, a fraction is unchanged in value if both numerator and denominator are multiplied by the same number. We choose a multiplier which will convert the denominator into a natural number—in this case we choose 100. In the accompanying display we see two things. First, the fraction $\frac{3.768}{.24}$ has the same value as the fraction $\frac{376.8}{24}$. The second thing which we notice in the display is that multiplication of a decimal number by 100 effectively moves the decimal point two places to the right from its original position. We have, then, that the problem $3.768 \div .24$ has the same answer as $376.8 \div .24$. The quotient is 15.7,

using the technique given earlier for locating the decimal point when the divisor is a natural number.

The step from this example to the general case is now easily taken, and it can be justified in exactly the same way—multiplication of numerator and denominator of a common fraction by a suitable multiplier. It follows that the mechanical rule is: in the division of numbers given in decimal form, move the decimal point in both divisor and dividend (an equal number of places and in the same direction) until the divisor is a natural number; perform the division, and align the decimal point in the quotient with the decimal point in the dividend. In moving the decimal point in the dividend, it may be necessary to append zeros in order to have a digit in each place value position of the numeral. Some examples are shown in Figure 7–10. In the last example, the arrows indicate the successive steps in the placement of the decimal point without having to write the problem more than once.

$.24\,\overline{)\,3.768} \longrightarrow 24\,\overline{)\,376.8}$, quotient 15.7

$.024\,\overline{)\,.003768} \longrightarrow 024.\,\overline{)\,003.768}$ or $24\,\overline{)\,3.768}$, quotient .157

$2.4\,\overline{)\,376.8} \longrightarrow 24.\,\overline{)\,3768.}$, quotient 157.

$.024\,\overline{)\,37.68} \longrightarrow 24.\,\overline{)\,37680.}$, quotient 1570.

$.24\,\overline{)\,.3768} \longrightarrow 24.\,\overline{)\,37.68}$, quotient 1.57

$2.4\,\overline{)\,3.768}$, quotient 1.57

FIGURE 7–10

Let us look at one more example. Divide 376.8 by 2400. The rule may be used to move the decimal points to the left as well as to the right.

$2400\,\overline{)\,376.8} \longrightarrow 24.00\,\overline{)\,3.768}$ $\quad 24\,\overline{)\,3.768}$, quotient .157

Other rules for determining the position of the decimal point in a quotient can be found in many arithmetic books. This might well be an opening for additional study.

NON-TERMINATING REPEATING DECIMALS

Previously in the section it was mentioned that the decimal representations for some fractions, for example, $\frac{1}{3}$, would not terminate. If the decimal representation is non-terminating but is repeating, then the decimal number does represent some ordinary common fraction. It might be of interest to look at such a problem now. Let us assume that in the decimal representation .54295429 . . . the group of digits 5429 is repeated infinitely. What common fraction is represented by this decimal representation? We will let the letter f represent this fraction: $f = .542954295429. \ldots$ We multiply both sides of this equation by 10,000 (this multiplier being chosen so that in the product the decimal will be placed just after the first group 5429).

$$10{,}000f = 10{,}000\,(.54295429\ldots) = 5429.54295429\ldots$$

We can see that the right-hand member still contains the fraction f. That is, $10{,}000f = 5429 + f$. Let us subtract f from each member of the equation: $10{,}000f - f = 5429$, or $9{,}999f = 5429$. When we divide both members of this last equation by 9,999, we find that the fraction f is $f = \frac{5429}{9999}$.

(a)	$f = .33\ldots$
(b) Multiply by 10	$10f = 3.33\ldots$
(c) Subtract (a) from (b)	$10f = 3.33\ldots$
	$f = .33$
	$9f = 3$
	$f = \frac{3}{9}$
	$f = \frac{1}{3}$

Before concluding, we should note that the subject of decimal fractions is far from exhausted. Three additional related topics which might serve as springboards for further study are:

(1) the techniques of "rounding,"

(2) the effects of rounding of intermediate data on the final result in each of the four fundamental operations, and

(3) scientific notation and significant digits.

EXERCISES

1. Write as decimal fractions:
   (a) Four hundred three thousandths
   (b) Four hundred and three thousandths
   (c) Three thousandths
   (d) Four hundred three hundred-thousandths

2. Change each decimal fraction to a common fraction.
   (a) .14 (c) 71.56002
   (b) 2.831 (d) 40.0001

3. Change each common fraction to a decimal notation.
   (a) $\frac{5}{12}$ (c) $\frac{7}{28}$ (e) $15\frac{3}{500}$
   (b) $\frac{9}{45}$ (d) $2\frac{3}{8}$

4. Write each of the following as a true number sentence.
   (a) $\frac{7}{8}=\frac{M}{56}$ (c) $\frac{5}{6}=\frac{20}{M}$ (e) $\frac{4}{4}=\frac{8}{M}$
   (b) $\frac{5}{3}=\frac{M}{15}$ (d) $\frac{7}{2}=\frac{M}{24}$ (f) $\frac{7}{3}=\frac{21}{M}$

5. Add, using the definition of the sum of fractions.
   (a) $\frac{1}{3}+\frac{1}{6}$
   (b) $\frac{1}{5}+\frac{3}{8}$

6. Add, using a least common denominator.
   (a) $\frac{1}{2}+\frac{3}{7}+\frac{3}{4}$
   (b) $\frac{3}{8}+\frac{7}{16}+\frac{1}{6}$

7. How did you determine the least common denominators in Exercise 6?

8. Add the fractions.

   (a) $\frac{3}{28}+\frac{2}{35}+\frac{6}{49}$

   (b) $\frac{1}{90}+\frac{1}{12}+\frac{7}{15}$

9. Subtract.

   (a) $\frac{5}{42}-\frac{8}{15}$

   (b) $\frac{10}{28}-\frac{11}{90}$

10. Multiply by use of the definition, then simplify by the use of equivalent fractions.

    (a) $\left(\frac{5}{42}\right)\cdot\left(\frac{8}{15}\right)$ (c) $\left(\frac{7}{8}\right)\cdot\left(\frac{16}{14}\right)$

    (b) $\left(\frac{10}{28}\right)\cdot\left(\frac{11}{90}\right)$ (d) $\left(\frac{48}{5}\right)\cdot\left(\frac{6}{13}\right)$

11. Divide by multiplying numerator and denominator by the reciprocal of the denominator.

    (a) $\dfrac{\frac{5}{8}}{\frac{7}{9}}$ (c) $\frac{16}{5}\div\frac{3}{4}$

    (b) $\frac{15}{16}\div\frac{1}{8}$ (d) $\frac{13}{14}\div\frac{2}{7}$

12. The common denominator method for dividing fractions involves expressing both the divisor and dividend with a common denominator and then dividing numerators. For example:

$$\frac{\frac{5}{8}}{\frac{7}{9}}=\frac{\frac{45}{72}}{\frac{56}{72}}=\frac{45}{56}$$

    Divide, using the common denominator method.

    (a) $\frac{15}{16}\div\frac{1}{8}$ (c) $\frac{13}{14}\div\frac{2}{7}$

    (b) $\frac{16}{5}\div\frac{3}{4}$

13. Add.

    (a) $.378 + 1.569 + 3.741$

    (b) $.68 + 1.70 + 5.43$

14. Subtract.

(a) $8.93 - 5.48$
(b) $697.0 - 52.6$

15. (a) Multiply: 43.2×9.57.
(b) Explain the positioning of the decimal point in the product.

16. Divide.

(a) $.54\overline{).6534}$ (c) $.0061\overline{)140.3}$
(b) $5.4\overline{)65.34}$ (d) $61\overline{).1403}$

17. Explain the common denominator method for dividing fractions.

18. Is the fraction $\frac{29}{41}$ equal to the fraction $\frac{31}{43}$? Discuss.

19. Some texts discuss addition and subtraction of "money numbers." Are these numbers not covered by standard arithmetic methods? Discuss reasons for and against their inclusion as separate entities.

20. Illustrate the use of a separatrix in the binary and quinary systems of numeration. Show how to convert a decimal fraction to a binary or quinary fraction.

21. Investigate the method used by the Egyptians for denoting fractions. Why is this more cumbersome than the numerals used today?

22. The Greeks believed in "unity" and did not utilize "fractions." Fractional concepts were discovered by the Pythegoreans who kept them a secret for many years. Investigate this phase of the history of numbers.

Select the appropriate response to each of the following statements and explain the reason for your choice.

23. If the denominator of a common fraction is made larger, what effect does this have upon the value of the decimal equivalent of the fraction?

(a) it is increased
(b) it is decreased
(c) there is no effect upon the value
(d) it is increased or decreased by powers of ten

24. If the numerator of a common fraction is made larger while the denominator is not altered, what effect does this have on the value of the fraction?

(a) it is increased
(b) it is decreased
(c) there is no effect upon the value
(d) it is increased or decreased by powers of ten

25. Which of the following would be another way of writing .010?

(a) $\frac{10}{100}$ (c) $\frac{1}{1000}$

(b) $\frac{10}{1000}$ (d) $\frac{1}{10}$

26. Which of the following would be another way of writing .100?

(a) $\frac{10}{100}$ (c) $\frac{1}{100}$

(b) $\frac{10}{1000}$ (d) $\frac{100}{100}$

27. What arithmetic principle from the list given below enables us to solve division problems involving fractions by the algorithm shown here?

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

(a) inverting the divisor and multiplying gives a correct result by definition.
(b) multiplying the numerator and denominator of a complex fraction by the denominator does not change the value of the fraction.
(c) the reciprocal of the denominator may be multiplied into both elements of a complex fraction without changing the value of the fraction.
(d) the reciprocal of the dividend may be multiplied by both elements of the divisor without changing the value of the fraction.

28. If both the numerator and denominator of a fraction are divided by one-half, the value of the resultant fraction is

(a) half as large.
(b) doubled in value.
(c) unchanged in value.

Extended Activities

1. Define a prime number.

2. The *Fundamental Theorem of Arithmetic* states that any integer (positive or negative "whole" number) may be factored into just one set of prime factors exclusive of the order and signs of the factors. Express each of these numbers as a product of prime factors.

(a) 16 (c) 35 (e) 210
(b) 28 (d) 90 (f) 84

3. The least common multiple of a set of positive "whole" numbers is the smallest non-zero number into which each may be divided (without re-

mainder). For example, the least common multiple of 16 and 28 may be found by comparing their prime factors:

$$16 = 2 \cdot 2 \cdot 2 \cdot 2 \qquad \text{l.c.m.} = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7$$
$$28 = 2 \cdot 2 \cdot 7$$

Find the l.c.m. of the numbers:

(a) 16, 35	(d) 35, 210	(g) 90, 12, 15
(b) 28, 35	(e) 16, 6, 9	(h) 8, 15, 42
(c) 28, 90	(f) 28, 35, 49	

4. Why were extended activities 2 and 3 given to extend mathematics understanding in Chapter 7?

5. Describe a situation in primitive life in which the natural numbers did not provide an adequate basis for the description of a quantitative situation.

6. (a) Show that $\frac{2}{3} < \frac{11}{12}$.

   (b) Then show that $\frac{2}{3} < \frac{2+11}{3+12}$ and that $\frac{2+11}{3+12} < \frac{11}{12}$.

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Chapter 8

Concepts to be developed in this chapter are:

1. *The understanding of the structure of our number system enables us to place elementary school mathematics in its larger context.*
2. *The rational numbers are those with which we usually work in the elementary school.*
3. *The number system was developed to keep pace with our need for new kinds of numbers.*

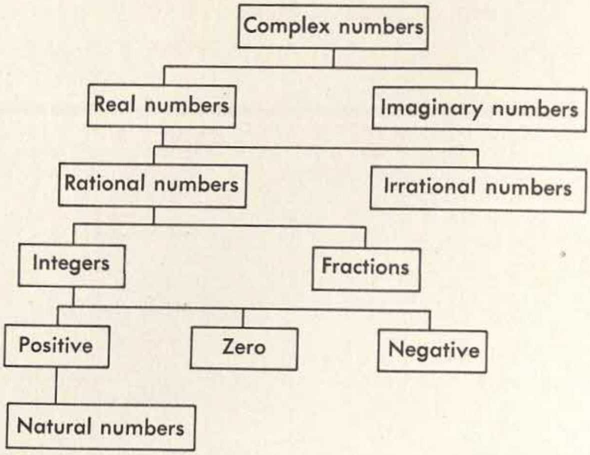
8

Understanding the Structure of the Number System

In this chapter we will show the relationship which exists between the development of the structure of our number system and the four fundamental operations. While much of the content of this chapter has been treated earlier, it is offered here as a comprehensive unit.

First, let us consider the term, "fundamental operations." It is used deliberately to describe what we have traditionally called the "fundamental processes." This terminology is more modern than that frequently used and refers to the four operations of addition, subtraction, multiplication, and division.

The following diagram of the structure of our number system includes all of the types of numbers which we are likely to encounter through the senior year in college. It is read from the bottom up, since both historically and structurally, the natural numbers come first.



THE NATURAL NUMBERS

The natural numbers are the counting numbers and are always considered to be positive. Before leaving the consideration of the natural numbers and moving up in the hierarchy, let us look at one possible way to classify them. In counting, we quite frequently are interested only in the number of objects, but sometimes we are interested in the order of the objects. For example, if we are hurrying to the ticket window at a movie theater, we might observe that there are three people already there. When we join the group there will be four people, but because of the order in the line, we are fourth in line. These two different kinds of counting numbers are called cardinal and ordinal, respectively. That is, the cardinals are concerned with magnitude, while the ordinal numbers are concerned with order. Some psychologists suggest that the sense of ordinal numbers preceded that of cardinal—that primitive man had the notion of first, second (coming after the first), etc., long before he ever made the abstraction of one object, two objects, etc. It is an interesting point on which to speculate, but it would be very difficult to prove or disprove, since, as far as we know, no society on earth ever developed one concept without the other.

ADDITION

In looking at the operation of addition, we see that any two *natural* numbers (counting numbers) may be paired in this operation and that

their sum is also a natural number. Hence, no new numbers are needed if addition is the only operation we want to perform.

NATURAL NUMBERS (COUNTING NUMBERS)

1, 2, 3, 4, 5, 6, 7, 8, 9, ...

Examples of Addition with Natural Numbers

$1 + 3 = 4$	$6 + 3 = 9$
$3 + 2 = 5$	$4 + 4 = 8$
$4 + 5 = 9$	$1 + 7 = 8$

SUBTRACTION

How do we define subtraction? At first, the definition must be restricted to the natural numbers since we assume they are the only ones we have in our system. We define $a - b$ to be the number c if (1) $a > b$ (read a is greater than b) and if (2) $a = b + c$. The first restriction is necessary in order for the second to have meaning for all natural numbers, for if b and c are natural numbers, then $b + c$ is greater than either, so a (which is $b + c$) must exceed b. The trouble with this definition, though, is that sometimes we may take all objects away from a group. Then, how many are left? None, of course, and we decide that zero is a good name and 0 a good symbol to stand for this number. That is, we find that if we want to generalize our definition of subtraction to include the case in which $a = b$, then we must append to our number system a new number, zero, not previously included in the set called natural numbers. A slight extra generalization leads us to try to let $a - b = c$ for *all* natural numbers a and b. To do this for *all* pairs a and b, we must have something to call the answer when it happens that $b > a$ (read b is greater than a, or b exceeds a). We now introduce into our system the negative numbers (quantitatively the same as the natural numbers, but with a negative sign or meaning). An illustration

Examples of Subtraction

	if (1) $a > b$	and if (2) $a = b + c$
$a - b = c$		
$6 - 3 = 3$	$6 > 3$	$6 = 3 + 3$
$9 - 2 = 7$	$9 > 2$	$9 = 2 + 7$
$4 - 1 = 3$	$4 > 1$	$4 = 1 + 3$

will serve to show how and why this is done. Let us consider the problem $7 - 12 = \boxed{}$. We find that the $\boxed{}$ must contain -5 in order that our whole system be consistent. Why? Because if $a - b = c$ means that $a = b + c$, then $7 - 12 = \boxed{}$ would need to mean that $12 + \boxed{} = 7$. If the box contains -5, and if $12 + \boxed{-5}$ is interpreted as $12 - 5$ which *is* equal to 7, the group of meanings of negative numbers and subtraction will be consistent with the meanings given under the first definition of subtraction for natural numbers a and b (with $a > b$).

We now identify the natural numbers as being positive (as opposed to negative in the above sense). With this identification, a new set of numbers is completed. This is the set of numbers called "integers" and includes the natural numbers, their negatives, and zero.

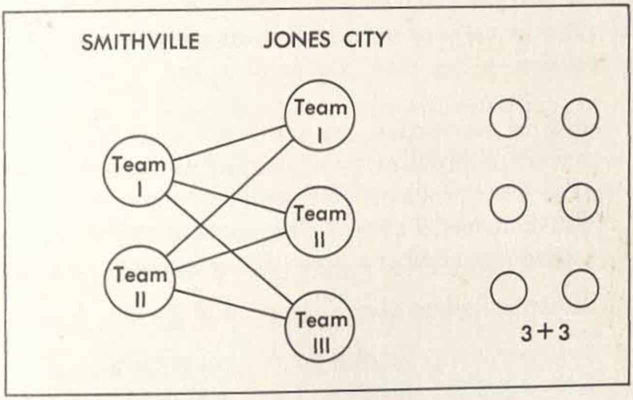
COMPLETENESS AND CLOSURE WITH RESPECT TO ADDITION AND SUBTRACTION

Among the integers, the operations of addition and subtraction may be applied to *any* pair and the result will also be an integer. Let us now consider two new words which have been adopted to describe these properties. First is the word "complete." An operation is described as complete (within a given set of numbers) if any two numbers of the set may be combined in this operation. This was the case for addition within the set of natural numbers, but was not the case for subtraction within the natural numbers (for we had to have $a > b$). Among the integers, though, we allow any two integers to be combined in the operation of subtraction (or addition). The other concept is that of "closure." A set is called "closed" under a certain operation if the *result* of combining two admissible numbers in the operation is also a number in the set. For addition and subtraction, the set of integers is both closed and complete, in that *any* two integers may be combined in these operations, and the result is also an integer. (An example of a set which is not closed under these two operations is the set of odd numbers. The sum or difference of two odd numbers is not itself an odd number, so the set of odd numbers is *not closed* with respect to the operations of addition or subtraction.)

MULTIPLICATION

The multiplication of natural numbers may be defined in terms of repetitive additions or in terms of the number of distinct pairings of

members of two sets. For example, 2×3 could represent the number of possible different team combinations which could be planned for games between two cities, one with two high school teams and the other with three. Also, 2×3 can represent the more familiar $3 + 3$. The latter definition may well be the more useful one in the elementary school.



In order to extend the definitions to include all of the integers, rather than natural numbers only, it is necessary to state what the product of any two integers would be. The possible multiplicative combinations of integers are shown in the following illustration.

RULES FOR MULTIPLYING INTEGERS

(1)	$(+a)(+b) = +(ab)$	Positive integer times a positive integer = positive integer
(2)	$a \cdot 0 = 0$	Any integer times zero = zero
(3)	$(-a)(+b) = -(ab)$	Negative integer times a positive integer = negative integer
(4)	$(+a)(-b) = -(ab)$	Positive integer times a negative integer = negative integer
(5)	$(-a)(-b) = +(ab)$	Negative integer times a negative integer = positive integer

Some of these properties may be *proved*, rather than given as definitions, if a proper set of axioms is assumed. However, we are approaching the problem from the standpoint of the development of the structure, rather than axiomatically.

It is doubtful that properties (3), (4), or (5) are useful in the ele-

mentary school, but they should be recognized as part of the complete structure. These same rules apply to the multiplication of all real numbers.

DIVISION

Division of integers is defined as follows: $a \div b = c$ means (1) $a = b \cdot c$, and (2) c is unique, and (3) c is an integer. This definition is obviously restrictive; for most values of a and b, c will not be an integer.

In division as in subtraction, we are faced with the choice of not being able to work all problems or of needing new numbers to describe our answers for some problems. For use in subtraction we introduced zero and negative numbers. Here we see another opportunity (or obligation) to expand our number system. If we remove the third restriction, we will have to admit that, $\frac{a}{b}$ (a symbol for $a \div b$) is a number even when a is not evenly divisible by b, as in the case of $3 \div 5$. These numbers are the common fractions. There was no need for them until the operation of division was introduced. To make the structure complete, we let the fraction $\frac{a}{1}$ have the value a (a quite normal choice).

One consequence of the definition ($a \div b = c$) which is not obvious, is that b cannot be zero. Why? By definition of division $a \div 0 = c$ would mean that $a = 0 \cdot c$. If $a \neq 0$, then this violates the multiplication property that $c \cdot 0 = 0$. If $a = 0$, then $0 \div 0 = c$ implies $0 = 0 \cdot c$, which is true for every integer c instead of being true uniquely as is required by (2) of our definition.

The integers and the fractions constitute the set of rational numbers. We seldom develop the structural chart beyond these concepts in the elementary grades.

FRACTIONS AND THE FUNDAMENTAL OPERATIONS

Having introduced the fractions, we now must go back and define addition, subtraction, multiplication, and division for them. Addition is defined as: $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$. This definition is chosen because it actually conforms to the way fractional parts of real objects may be combined.

We also need to define the concept, the equality of fractions: $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$. This definition now allows us to assert that $\frac{a}{b} = \frac{ac}{bc}$ since $abc = acb$ (using the commutative property). This last result is the one which enables us to simplify, or reduce fractions which have a common factor in both numerator and denominator.

Multiplication is defined as $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. For example, $\frac{5}{9} \cdot \frac{2}{11} = \frac{10}{99}$. When this definition is combined with the definition for the equality of fractions given above, there is a surprising result. This result is that for every non-zero rational number, there exists what is called a multiplicative inverse. That is, for each number a, if $a \neq 0$, there exists a number b for which $a \cdot b = 1$. What is the number b? It is $\frac{1}{a}$, another rational number. What is the inverse (or reciprocal) of the fraction $\frac{a}{b}$? It is $\frac{b}{a}$. Why? Because $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = \frac{1}{1} = 1$.

Now, we are in a position to answer a question which we have been asked (and which is asked every year in almost every sixth-grade classroom). Why, in dividing fractions, do we invert and multiply? In the light of the above work, this can be answered as follows. Let us solve the problem $\frac{5}{9} \div \frac{2}{11}$. Let a stand for the answer. That is, $a = \frac{5}{9} \div \frac{2}{11} = \frac{\frac{5}{9}}{\frac{2}{11}}$. For each fraction there is an inverse, and for $\frac{2}{11}$ the inverse is $\frac{11}{2}$. Also, for each fraction $\frac{a}{b}$, $\frac{a}{b} = \frac{ac}{bc}$, so

$$\frac{\frac{5}{9}}{\rightarrow \frac{2}{11}} = \frac{\frac{5}{9} \cdot \frac{11}{2}}{\frac{2}{11} \cdot \frac{11}{2}} = \frac{\frac{5}{9} \cdot \frac{11}{2}}{1} = \frac{5}{9} \cdot \frac{\overset{\downarrow}{11}}{2} = \frac{55}{18}$$

We can see that the arrows point to $\frac{2}{11}$ as a divisor and $\frac{11}{2}$ as a multiplier. That is, as a consequence of our definitions of the operations of multiplication and division, and of the equality of fractions, the divisor was inverted and now appears as a multiplier.

Further progress through the chart of number sets given at the outset

of the chapter would require further definitions of addition, multiplication, subtraction, division, and equality.

THE IRRATIONALS

The structure chart shows that the rational numbers are composed of the integers and the fractions. The irrationals are those real numbers which are neither integers nor fractions.

Probably the best known example of an irrational number is $\sqrt{2}$. The symbol $\sqrt{2}$ stands for a number whose square is 2. That is, $\sqrt{2} \times \sqrt{2} = 2$. Man's first encounter with this number was in antiquity, and we still encounter it today. For example, it occurs as the length of the diagonal of a square whose sides are each one unit long. The Pythagoreans, the followers of the Greek philosopher and mathematician Pythagoras, were so shocked to find that such a number exists that they concealed its discovery for many years.[1]

Another irrational number with which we are familiar is the number π which we encounter when finding the area and circumference of a circle. Other such numbers exist in great profusion. But for all of our powers, we still have not been able to show whether some numbers are rational or irrational.

REAL AND IMAGINARY NUMBERS

As the chart shows, the rationals and irrationals taken together constitute the real numbers. And as the structure chart shows, the reals are only a part of an even more extensive type of number.

There are several ways in which we can approach the subject of imaginary numbers. The older approach is to define an imaginary unit i to be $i = \sqrt{-1}$. We can see why this would be called imaginary, since we cannot find a number among those with which we are familiar whose square is negative. There is a newer approach which is more satisfactory from the standpoint of treating the properties of the number system more rigorously. Because of space limitations, we will simply say that in this newer approach, the imaginary numbers may be associated with points on a line which is perpendicular to the line

[1] A very readable account of this episode of history may be found in Tobias Dantzig, *Number: The Language of Science* (New York: The Macmillan Co., 1954.)

of real numbers. The unit distance on this line is associated with the number i, and all of the points of the line are associated with the imaginary numbers, in a similar fashion to the association used for the real numbers.

COMPLEX NUMBERS

Once the imaginary numbers have been introduced, the complex numbers may be presented. The traditional approach is to let a complex number be of the form $a + ib$, in which a and b are real numbers. The modern approach is to define a complex number as being an ordered pair of numbers (a, b) obeying certain rules of operation. The term "ordered pairs" means that the first number of the pair always means one thing and the second number another. Ordered pairs may be associated with all the points of the plane which is determined by the real and imaginary number lines by letting the first of the two numbers be the real component, and the second number the imaginary component of the position of the point. Of course, rules for adding and multiplying these new types of numbers must be stated before the properties of the number system may be investigated. We will not dwell further on complex numbers except to say that the real numbers with which we are familiar are special kinds of complex numbers in which the imaginary component of the number is zero.

EXERCISES

1. Write five consecutive natural numbers.
2. Write five consecutive odd negative integers.
3. (a) If the sum of two numbers is zero, what is known about the numbers?
   (b) If the product of two numbers is zero, what is known about the numbers?
4. (a) Define an even number.
   (b) Define an odd number.
5. Can a natural number be divided by a negative integer? Why or why not?

6. To which set(s) of numbers do these numbers belong?

(a) $+3$ (c) -3 (e) $\sqrt{4}$

(b) $\frac{5}{4}$ (d) $\frac{9}{3}$

7. To which set(s) of numbers do the prime numbers belong?

8. Find all the primes between 10 and 75.

9. Multiply 1.414 by itself. How close to 2 is your product? Try it with 1.4142 and 1.4143. To what irrational number is 1.414 approximately equal?

10. Are the natural numbers actually distinct from the positive integers? Discuss.

11. Diagram the structure of the number system (through the real numbers) to show that fractions and irrational numbers can be either positive or negative.

Select the appropriate response to each of the following statements and explain the reason for your choice.

12. In man's development of a system of numeration, which of the following came last?

(a) one-to-one correspondence
(b) place value
(c) numerals
(d) zero

13. The expression $7 - 9$ is meaningless in the set of

(a) integers
(b) natural numbers
(c) real numbers
(d) rational numbers

14. If a and b are natural numbers, which of these conditions if fulfilled would cause the expression $a \div b$ always to represent a natural number?

(a) $a > b$; a and b are even numbers
(b) $a = b$; a and b are not equal to zero
(c) $b > a$; a is an even number
(d) $a \neq b$; b is not equal to zero

15. Negative numbers

(a) do not appear in concrete situations
(b) were devised because division could not be performed without them

(c) cause the number systems to be closed and complete with respect to the subtraction operation
(d) are really not numbers at all

16. The expression $\frac{2}{3}$, as a symbol representing a number, is meaningless in the set of:

(a) real numbers
(b) rational numbers
(c) fractions
(d) integers

17. Which of the following choices is true?
Zero

(a) is a number
(b) is a natural number
(c) causes the number system to be closed and complete with respect to the subtraction operation
(d) is none of the above

Extended Activities

1. (a) Is the set of even numbers closed with respect to addition?
(b) Is the set of even numbers less than 10 closed with respect to addition? Explain your answers.

2. (a) Is the set of odd numbers closed with respect to addition?
(b) Is the set of odd numbers closed with respect to multiplication? Explain your answers.

3. Show that $\sqrt{2}$ is irrational.

4. (a) Find the meaning of "ordinal" and "cardinal" as applied to numbers.
(b) Which set(s) of numbers can be modified by the adjectives "ordinal" and "cardinal"?

5. Which of the following numbers are ordinal? Which are cardinal? Can any of them be either?
(a) Joe has 5 pencils.
(b) My office number is 407.
(c) This is page 24.
(d) It is 9:15 o'clock.
(e) There are 1768 seats in this auditorium.
(f) Abraham Lincoln was born in 1809.
(g) There were 25 people at the luncheon.

6. The sum of two rational numbers is rational. Can a similar statement be made about irrational numbers? Explain.

7. Using a number line, illustrate the number system through the set of real numbers.

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Chapter 9

Concepts to be developed in this chapter are:

1. *Equality is a relationship possessing certain definite properties.*
2. *An equal sign should be used only when a relation possesses these properties.*
3. *The nature of equivalence relationships needs to be understood in arithmetic.*

9

Refining the Use of Equivalence Relations

Teachers sometimes see such statements as these on the papers their pupils hand in: (1) John = 86 lbs., (2) Mary = 63 lbs., (3) Flagpole = 60 ft., (4) 2 cans beans = 25¢. Do students really mean equal when they use the equal sign, or do they mean something entirely different? Do we ever use the word equal or the symbol = improperly? It is the purpose of this chapter to discuss the meaning of equality with the hope that new understandings will be developed for those who have not previously considered these questions.

Yet another question should be asked before the topic is fully launched. Is equality something that must be defined; is it a relation which is intuitively understood by all and hence needs no definition; or is it merely an undefined term? Each of us must decide what the answer to this question will be before we can proceed.

First, let us consider the last of the three possibilities. In any system there must be some undefined terms, or else circular reasoning will

result. A prime example of circularity is contained in a dictionary which defines lovely as beautiful and beautiful as lovely. To a person who knows the meaning of neither word, the circular definition sheds no light whatever. In the case of "equal" or "equality" any definition which might be made would certainly in the final analysis need to depend on some undefined terms, so we might ask, "Why not let equality itself be undefined?" The answer to this question is that equality is a relation of sufficient complexity that its meaning may be expressed as a combination of simpler undefined terms.

The examples given at the beginning of the chapter, and others that all of us have seen, certainly justify the assertion that it is not true that the meaning of equality is intuitively understood by all and hence needs no definition. Other examples which also dispel the intuitive understanding hypothesis will appear before the end of the chapter.

The first of the three possibilities given in the question asked earlier represents the most realistic approach to equality. It is a relation which must be defined, and once it has been defined we should no more use the word "equal" improperly than we should say, "The boy laid on the bed."

EQUIVALENCE RELATIONS

Before giving a full definition of equality, let us turn our attention to what mathematicians call equivalence relations. Briefly, an equivalence relation is a relation between elements of a set, the relation being reflexive, symmetric, and transitive. Stated in symbols, using the symbol $\sim$ to stand for the relation between the elements, we say: if for all elements a, b, c, of a given set, it is true that

(1) $a \sim a$,
(2) if $a \sim b$, then $b \sim a$,
and (3) if $a \sim b$ and $b \sim c$, then $a \sim c$,
then $\sim$ is an equivalence relation.

Two comments are in order before any illustrations are given. One is that any symbol might have been chosen to denote the relation between elements. We have used $\sim$ for the symbol in the statements above. In the illustrations which follow we use the letter R to stand for a relation between elements. This is done not for confusion, but to illustrate that any symbol may be used. The second comment relates to the way in which we read statements like $a \sim b$ or aRb. The best way to read such statements is, "a relation b." If the relation is specified, then the given

relation might be read instead of the word "relation." For example, if R (or $\sim$) stands for the relation "is the brother of," then aRb (or $a \sim b$) would be read "a is the brother of b." Further examples are given in the work which follows.

It might be of interest to note that the three conditions which determine an equivalence relation are independent. To see this, we need only to show that there exist relations for which two of the conditions are satisfied while the third is not.

Let us first consider a relation that satisfies conditions (1) and (2) but fails to satisfy (3). If R refers to the relation between colors (elements of the set of all colors) which blend to give a harmonious visual impression, and if we let a represent a given shade of red, b represent a given shade of blue, and c represent a given shade of orange, the three conditions have the following meanings:

(1) "aRa" means "red blends harmoniously with red."
(2) "If aRb, then bRa" means "if red blends harmoniously with blue, then blue blends harmoniously with red."
(3) "If aRb and bRc, then aRc" means "if red blends harmoniously with blue and blue blends harmoniously with orange, then red blends harmoniously with orange."

We see that the first two conditions are satisfied for this particular relation, but the third is not satisfied since it is not necessarily true that if red blends with blue and blue blends with orange, red must blend with orange.

Let us consider a second example, discussed in slightly less detail. If we let the relation symbolized by R be the sibling relationship (not including half sibling), then the three conditions would read as follows:

(1) a is the sibling of a,
(2) if a is the sibling of b, then b is the sibling of a,
and (3) if a is the sibling of b and b is the sibling of c, then a is the sibling of c.

We can see that conditions (2) and (3) are satisfied, but that (1) fails to be satisfied since no one is the sibling of himself.

Finally, let us consider the relation "is greater than or equal to." If we let R symbolize this relation, then

(1) "aRa" means "a is greater than or equal to a."
(2) "If aRb, then bRa" means "if a is greater than or equal to b, then b is greater than or equal to a."

(3) "If aRb and bRc, then aRc" mean "If a is greater than or equal to b and b is greater than or equal to c, then a is greater than or equal to c."

In this example, conditions (1) and (3) are satisfied (for this relation), but condition (2) fails to be satisfied. In this particular illustration we might have used the common symbol $\geqq$ instead of R. Then the statement aRb would have been written as "$a \geqq b$" and would still have been read as "a is greater than or equal to b."

We might now ask the question, "Does there exist any relation other than the one we intuitively accept for equality which possesses these three properties?" Let us consider relation R which means "has the same shape as." Then the statements that (1) aRa, and (2) if aRb, then bRa, and finally (3) if aRb and bRc then aRc are all true. We see then that the conditions we use to define an equivalence relation do not uniquely determine the relation we would like to call equality. What other properties do we want "equal" to possess?

THE PROPERTIES OF EQUALITY

We will say that equality as a relation between real numbers possesses the following five properties (the first three of which cause it to be an equivalence relation). In the statement of these properties we use the symbol $=$ to denote the relation.

I. $a = a$. (Reflexive property)
II. If $a = b$, then $b = a$. (Symmetric property)
III. If $a = b$ and $b = c$, then $a = c$. (Transitive property)
IV. If $a = b$ and $c = d$, then $a + c = b + d$.
V. If $a = b$ and $c = d$, then $ac = bd$.

The last two of these relations may be paraphrased by saying, "If equals are added to equals, the sums are equal," and "If equals are multiplied by equals, the products are equal."

Do these properties characterize the relation we have always thought of as equality? For example, do these properties imply that equals may be substituted for equals in expressions containing sums and products? This is certainly a property we all expect of the relation we call equality. Let us look into this question in some detail by considering the following problem.

If $x = u$, $y = v$, $z = w$, is it permissible to substitute for x, y, and z in the expression $x^2z + y + yz$ to obtain an equal expression $u^2w + v$

$+ vw$? That is, is it true that $x^2z + y + yz = u^2w + v + vw$? First, using property V above and $x = u$,

$$x \cdot x = u \cdot u, \text{ or}$$
$$x^2 = u^2.$$

Using property V again and $z = w$, we obtain

$$x^2z = u^2w.$$

Using property IV with this last result and $y = v$,

$$x^2z + y = u^2w + v.$$

From property V and $y = v$, $z = w$, we have $yz = vw$, the elements of which, by property IV, may be added to the left and right members of the equation $x^2z + y = u^2w + v$ to obtain

$$x^2z + y + yz = u^2w + v + vw.$$

That is, in an expression involving sums and products of real numbers, equals may be substituted for equals.

In some characterizations of equality, properties IV and V above are replaced with the sum and product substitution properties, and then the properties we have called IV and V are proved as theorems.

Whichever set of properties we adopt, we are adopting a set of axioms. The axioms we adopt are basic assumptions which we make about the nature of the elements of our sets and the relations between these elements. We cannot prove our assumptions are true, but neither can we prove them false, for to do so we would have to rely on other assumptions. Axioms are not self-evident truths, though, as the ancient Greeks assumed, for frequently they are assumptions so bizarre as to preclude being self-evident, and yet they may lead to perfectly valid (logical) conclusions.

CONCLUSION

We now return to the statements at the beginning of the chapter for an examination of them in the light of the mathematical meaning of equality. Consider the statements, "John = 86 lbs.," "Flagpole = 60 ft.," and "2 cans beans = 25¢." What might the student have been thinking when he made these statements? Could he have meant that "John" and "86 lbs." are two different names for the same quantity? Probably not. In a statement like "Abraham Lincoln = The Great Emancipator," we might mean that these were two different names for the same person, but even in this situation we cannot always substitute one of these names for the other. Let us consider the assertion, "Thomas and Nancy Lincoln named their son Abraham Lincoln." Can we just as well substitute "The Great Emancipator" for "Abraham Lincoln" in this

assertion? If we agree that we would like to be able to substitute a quantity for its equal, we would have to say, then, that in "Abraham Lincoln = The Great Emancipator" the equal sign is improperly used. Hence, we conclude that we do not use an equal sign merely to indicate the relation between two names for the same object or person.

In the statement "2 cans beans = 25¢" we obviously were not referring to two different names for the same object. The reference really was to the *number* of cents in the cost of two cans of beans being the same as the number of cents in 25¢. That is, all of the statements attributed to children were attempts to express relations between numbers. It is important that we encourage children to make attempts to express numerical relations, but at the same time we should guide them toward an understanding of the equality relation in order to enable them to make their statements properly, and thereby convey the meaning they intend to convey.

EXERCISES

1. Is any of the following a correct statement of equality? Why?
   (a) 62 lbs. = John's weight
   (b) 1 gal. white paint = \$4.65
   (c) 1 teacher = 30 pupils
   (d) 1 mile = 5280 feet

2. Is any of the following an incorrect statement of equality? Why?
   (a) $12 \times 12 = 140 + 4$
   (b) 1 yard = 36 inches
   (c) $2 + 2 = 4$
   (d) four quarters of butter = 1 pound

3. Solve the equation $3x + 4 = 2x + 7$ for the value of x. Explain what property of equality is used at each step of your solution.

4. If $x = a^2$, $y = 3ab$, show that $x^2 + y + y^2 = a^4 + 3a^3b + 9a^2b^2$.

5. Show which of the following relations are equivalence relations:
   (a) geometric symmetry
   (b) geometric similarity (denoted by $\approx$)
   (c) geometric congruence (denoted by $\cong$)

6. Show why, if $x = u$, $y = v$, that $\dfrac{x^2 - 3xy}{y^2} = \dfrac{u^2 - 3uv}{v^2}$. Explain why it is necessary to require that $y \neq 0$.

7. Show why these relations are not equivalence relations:
   (a) less than or equal
   (b) less than
   (c) greater than or equal
   (d) greater than

8. Describe a relation (and the set of elements related) which is symmetric and reflexive but not transitive.

9. Describe a relation (and the set of elements related) which is symmetric and transitive but not reflexive.

10. Describe a relation (and the set of elements related) which is reflexive and transitive but not symmetric.

Extended Activities

1. Two integers are said to be *relatively prime* if they have no common whole-number factor greater than 1. Show that the relation "is relatively prime to" is not an equivalence relation by showing which of the following statements are false:
   (a) a is relatively prime to a
   (b) If a is relatively prime to b, then b is relatively prime to a
   (c) If a is relatively prime to b, and b is relatively prime to c, then a is relatively prime to c

   You may show any statement to be false by finding a single example for which it is false.

2. In a manner similar to that used in Extended Activity 1, show why the relation "is an integral multiple of" is not an equivalence relation.

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Solutions for Selected Exercises

(*Answers to exercises which require subjective responses and to Extended Activities are not given*)

Chapter 1

1. (a) True. The numeral "3 + 4" is a numeral for the number 7.
   (b) True. You can put the numeral "7" on the chalkboard.
   (c) False. The numeral "7" is not a number and hence cannot be an odd number.
   (d) True. The number 9 is divisible by the number 3.
   (e) False. The number 7 is not a numeral, so it cannot be a numeral for something.
   (f) False. The numeral "3" cannot be added to the numeral "4"–at least not meaningfully. We add numbers, not numerals.
   (g) False. There are no numerals "4" in the number 32.

2. (a) Insert no quotation marks.
   (b) "15" is a name for the number 5 + 10.
   (c) The "5" you wrote on the chalkboard is poorly formed.
   (d) You can subtract 5 from 12, but you cannot subtract "5" from "12".

3. They are a numeral; they denote a number.
   They may then merely be the set of four fingers and may not denote the number of the set.

5. They probably mean two seats are available. They may ask the question, "Are there two of you?" Since they stand for number, they are probably a form of numeral. They can be put into one-to-one correspondence with two vacant seats, the two people, and, more obscurely, with the two people's heads, etc.

6. An unlimited number. In practical use there may be only a few like 8, VIII, III, etc., but the number of symbols which *can* be used is unlimited.

8. No. It is necessary also to show that there is only one finger for each chair leg to make the one-to-one correspondence complete. If there are five fingers to four legs, there is not a one-to-one correspondence.

10. By showing the coconuts themselves.
    To show 100 other concrete objects–say, seashells–the number of which has been determined only insofar as there is a one-to-one correspondence.

    Use the correct number of semi-concrete objects, like tally marks, to denote the 100 coconuts. Use a numeral like 100 (which, in itself, has no property of one-hundredness) to represent them.

11. (a) Yes. Written and spoken words. Red light for stop. Abbreviations. ONE WAY . Sound of bell to denote time to start or end class.

    (b) No.

12.
```
X X X X X
| | | |
o o o o
```

Chapter 2

1. One such correspondence is: a–1
   b–2
   c–3
   d–4
   e–5

2. "a" can correspond to any of 5 numerals; then "b" can correspond to any of the 4 remaining numerals; then "c" can correspond to any of the 3 remaining numerals; etc. The answer: $5 \times 4 \times 3 \times 2 \times 1 = 120$.

3. $A \cap B = \{b, d, e, a\}$, $A \cup B = \{a, b, c, d, e, f, g, h, j, k\}$
   $(A \cup B) - (A \cap B) = \{c, f, g, h, j, k\}$

7. (a) $\{A \cap \underline{B}\} = \{c, d, e, f\}$, so $n(A \cap B) = 4$.
   (b) $\{A \cup \widetilde{C}\} = \{a, b, c, d, \ldots, p, q, r\}$, so $n(A \cup \widetilde{C}) = 18$.

11. (a) $\{1, 2, 3\}$
    (b) $\{1, 2, 3\}$
    (c) ϕ
    (d) ϕ
    (e) $\{6\}$

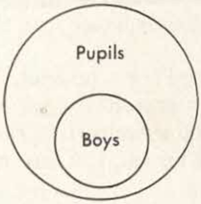
12. (a) 3
    (b) 3
    (c) 0
    (d) 0

14. No. Look at #12 (b) compared with #13 (b).

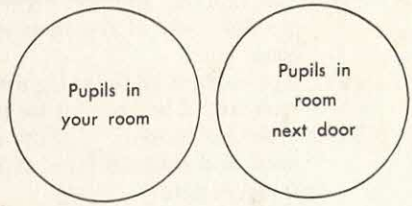
17. (c), (d)

18. (b) and (c).

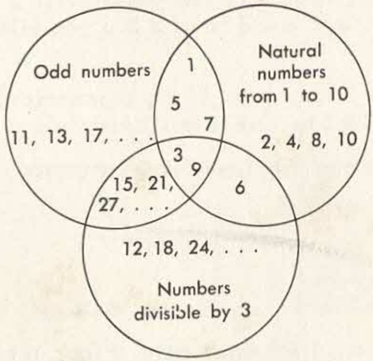
19. (a)



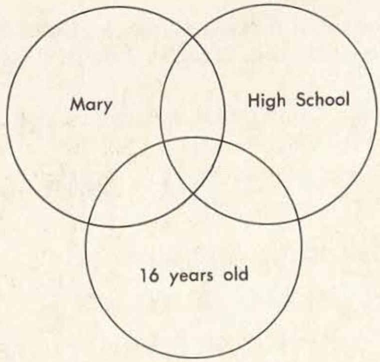
(b)



(c)



(d)



This is not the only possible diagram.

Chapter 3

1. "hundred" means "ten tens"
   "thousand" means "ten hundreds" or "ten tens of tens"

3. (a) 3 eighty-ones, 4 nines, 7 ones
   (b) 2 sixty-fours, 1 sixteen, 2 fours, 1 one
   (c) 7 five-hundred twelves, 3 sixty fours, 5 eights, 2 ones

5. Five: 0, 1, 2, 3, 4.

7. (a) 955, 310; (b) 0, or, if all digits must be used and decimal fractions are allowed, .013559.

8. (a) Our modern addition algorithm can be used in other bases than ten, but only if the facts of addition are expressed in the other base, too.
   (b) Our modern addition algorithm is designed for a positional system, so it would be fruitless to argue whether it would work in a non-positional system. In many non-positional systems it could not be used, and some features of it would not be valid in any non-positional system.
   (c) The use of Hindu-Arabic numerals per se is not a feature of our modern algorithm, so it could be used in a non-Hindu-Arabic positional system.

9. The cost of the oranges is nine cents. If 21¢ represents nine cents, then the base must be four. That is, 2 fours and 1 make nine.

10. The pocket being described is the one which represents the square of the base.
    (a) four
    (b) forty-nine
    (c) one hundred forty-four
    (d) two hundred fifty-six

11. 57_{ten} contains 1 thirty-two, 1 sixteen, 1 eight, 0 four, 0 two, 1 unit—the numeral then is 111001_{two}

    391_{ten} contains 1 two hundred fifty-six, 1 one hundred twenty-eight, 0 sixty-four, 0 thirty-two, 0 sixteen, 0 eight, 1 four, 1 two, 1 unit—the numeral is 110000111_{two}

    126_{ten} contains 1 sixty-four, 1 thirty-two, 1 sixteen, 1 eight, 1 four, 1 two, 0 units—the numeral is 1111110_{two}

13. (a) 1001, (b) 10101, (c) 11111, (d) 1000001, (e) 1100100, (f) 10000000

15. (a) 2775, (b) 101011010111, (c) 42100

17. (a) 4, (b) 12, (c) 4, (d) 34, (e) 122

19. (a) 120, (b) 242, (c) 443, (d) 2404, (e) 1000

21.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

(a) 14
(b) 13
(c) 24
(d) 11
(e) 3
(f) 27

23. (a) The terminal digit of even numbers is zero, for odd numbers is 1
(b) If the last two digits are 00.

25. $10{,}000{,}000{,}000_{\text{seven}}$

27. (a) 1.492×10^3
(b) 6×10^6

28. (a)

$$\begin{array}{r} 391 \\ +\ 57 \\ \hline 448_{\text{ten}} \end{array} \qquad \begin{array}{r} 3031 \\ +\ 212 \\ \hline 3243_{\text{five}} \end{array} \qquad \begin{array}{r} 607 \\ +\ 71 \\ \hline 700_{\text{eight}} \end{array} \qquad \begin{array}{r} 110000111 \\ +\ \ 111001 \\ \hline 111000000_{\text{two}} \end{array}$$

(b)

$$\begin{array}{r} 391 \\ -126 \\ \hline 265_{\text{ten}} \end{array} \qquad \begin{array}{r} 3031 \\ -1001 \\ \hline 2030_{\text{five}} \end{array} \qquad \begin{array}{r} 607 \\ -176 \\ \hline 411_{\text{eight}} \end{array} \qquad \begin{array}{r} 110000111 \\ -\ \ 1111110 \\ \hline 100001001_{\text{two}} \end{array}$$

(c)

$$\begin{array}{r} 126 \\ \times\ 57 \\ \hline 882 \\ 630 \\ \hline 7182_{\text{ten}} \end{array} \qquad \begin{array}{r} 1001 \\ \times\ 212 \\ \hline 2002 \\ 1001 \\ 2002 \\ \hline 212212_{\text{five}} \end{array} \qquad \begin{array}{r} 176 \\ \times\ 71 \\ \hline 176 \\ 1562 \\ \hline 16016_{\text{eight}} \end{array} \qquad \begin{array}{r} 1111110 \\ \times\ 111001 \\ \hline 11111110 \\ 1111110 \\ 1111110 \\ 1111110 \\ \hline 1110000001110_{\text{two}} \end{array}$$

29. (using T for ten and E for eleven)

Base Ten	Base Five	Base Three	Base Twelve
1	1	1	1
2	2	2	2
3	3	10	3
4	4	11	4
5	10	12	5
6	11	20	6
10	20	101	T
11	21	102	E
12	22	110	10
16	31	121	14
21	41	210	19
25	100	221	21
26	101	222	22

30. 0, 1, 2, 3

32.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

33.

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	10	12
3	0	3	12	21

Chapter 4

1. You used the Associative Property of Addition to allow adding some two of the four numbers first, and you used the Commutative Property of Addition to allow interchange of order.

3. Yes. $(.5 + .2) + .4 = .5\ (.2 + .4)$

4. Yes. Commutativity is not a property of the positional system of numeration but of multiplication (and addition).

6. $3 - (5 - 2) \neq (3 - 5) - 2$; $16 \div (8 \div 2) \neq (16 \div 8) \div 2$.

7. $5 \times (4 - 2) = 5 \times 4 - 5 \times 2$; $(8 + 4) \div 2 = (8 \div 2) + (4 \div 2)$; $(8 - 4) \div 2 = (8 \div 2) - (4 \div 2)$.

9. $5 \times 7 = 5 \times (4 + 3) = (5 \times 4) + (5 \times 3)$, each term of which is known.

10. Distributive. Actually, the division by 6 may be expressed as multiplication by 1/6, and the multiplication is distributive with respect to addition.

11. (a) $369 \div 3 = (300 + 60 + 9) \div 3 = (300 \div 3) + (60 \div 3) + (9 \div 3)$
    (b) $4832 \div 16 = (4800 \div 16) + (32 \div 16)$

13. and 14.

$$\begin{aligned}23 \times 47 &= 23 \times (40 + 7) = (23 \times 40) + (23 \times 7)\\ &= (20 + 3) \times 40 + (20 + 3) \times 7\\ &= 20 \times 40 + 3 \times 40 + 20 \times 7 + 3 \times 7\end{aligned}$$

(a)
$$\begin{array}{r} 23 \\ \underline{\times 47} \\ 21 \\ 140 \\ 120 \\ \underline{800} \\ 1081 \end{array}$$

(b)
$$\begin{array}{r} 23 \\ \underline{\times 47} \\ 161 \\ \underline{92} \\ 1081 \end{array}$$

15. In the work that follows, the following code will be used to state the justification for each step:

CA – Commutative property of addition
AA – Associative property of addition
CM – Commutative property of multiplication
AM – Associative property of multiplication
D – Distributive property of multiplication, with respect to addition. Both left-hand and right-hand distribution will be used–i.e., $a(b + c) = ab + ac$ or $(b + c)\ a = ba + ca$.
P – Some property of positional notation
FA – Some fact of addition
FM – Some fact of multiplication

The following represents one way to justify the multiplication:

$$\begin{aligned}27 \times 43 &= (20 + 7) \times 43, \text{P}\\ &= 20 \times 43 + 7 \times 43, \text{D}\\ &= 20 \times (40 + 3) + 7 \times (40 + 3), \text{P}\\ &= [20 \times 40 + 20 \times 3] + [7 \times 40 + 7 \times 3], \text{D}\\ &= 20 \times 40 + 20 \times 3 + 7 \times 40 + 7 \times 3, \text{AA}\end{aligned}$$

$= (2 \times 10) \times (4 \times 10) + (2 \times 10) \times 3 + 7 \times (4 \times 10) + 7 \times 3$, P
$= 2 \times 10 \times 4 \times 10 + 2 \times 10 \times 3 + 7 \times 4 \times 10 + 7 \times 3$, AM
$= 2 \times (10 \times 4) \times 10 + 2 \times (10 \times 3) + (7 \times 4) \times 10 + 7 \times 3$, AM
$= 2 \times (4 \times 10) \times 10 + 2 \times (3 \times 10) + (7 \times 4) \times 10 + 7 \times 3$, CM
$= 2 \times 4 \times 10 \times 10 + 2 \times 3 \times 10 + (7 \times 4) \times 10 + 7 \times 3$, AM
$= (2 \times 4) \times (10 \times 10) + (2 \times 3) \times 10 + (7 \times 4) \times 10 + 7 \times 3$, AM
$= (8) \times (10 \times 10) + (6) \times 10 + (28) \times 10 + 21$, FM
$= 8 \times (10 \times 10) + 6 \times 10 + (20 + 8) \times 10 + (20 + 1)$, P
$= 8 \times (10 \times 10) + 6 \times 10 + (20 \times 10 + 8 \times 10) + (20 + 1)$, D
$= 8 \times (10 \times 10) + 6 \times 10 + 20 \times 10 + 8 \times 10 + 20 + 1$, AA
$= 8 \times (10 \times 10) + 6 \times 10 + (2 \times 10) \times 10 + 8 \times 10 + 2 \times 10 + 1$, P
$= 8 \times (10 \times 10) + 6 \times 10 + 2 \times 10 \times 10 + 8 \times 10 + 2 \times 10 + 1$, AM
$= 8 \times (10 \times 10) + 6 \times 10 + 2 \times (10 \times 10) + 8 \times 10 + 2 \times 10 + 1$, AM
$= 8 \times (10 \times 10) + [6 \times 10 + 2 \times (10 \times 10)] + 8 \times 10 + 2 \times 10 + 1$, AA
$= 8 \times (10 \times 10) + [2 \times (10 \times 10) + 6 \times 10] + 8 \times 10 + 2 \times 10 + 1$, CA
$= 8 \times (10 \times 10) + 2 \times (10 \times 10) + 6 \times 10 + 8 \times 10 + 2 \times 10 + 1$, AA
$= [8 \times (10 \times 10) + 2 \times (10 \times 10)] + [6 \times 10 + 8 \times 10 + 2 \times 10] + 1$, AA
$= [8 + 2] \times (10 \times 10) + [6 + 8 + 2] \times 10 + 1$, D
$= [8 + 2] \times (10 \times 10) + [(6 + 8) + 2] \times 10 + 1$, AA
$= [8 + 2] \times (10 \times 10) + [(8 + 6) + 2] \times 10 + 1$, CA
$= [8 + 2] \times (10 \times 10) + [8 + 6 + 2] \times 10 + 1$, AA
$= [8 + 2] \times (10 \times 10) + [8 + (6 + 2)] \times 10 + 1$, AA
$= [10] \times (10 \times 10) + [8 + 8] \times 10 + 1$, F
$= (10) \times (10 \times 10) + (16) \times 10 + 1$, F
$= 10 \times 10 \times 10 + 16 \times 10 + 1$, AM
$= 10 \times 10 \times 10 + (10 + 6) \times 10 + 1$, P
$= 10 \times 10 \times 10 + [(10 + 6) \times 10] + 1$, AA
$= 10 \times 10 \times 10 + [10 \times 10 + 6 \times 10] + 1$, D
$= 10 \times 10 \times 10 + 10 \times 10 + 6 \times 10 + 1$, AA
$= 1161$, P

19. None. All are appropriate. An analysis like that for problem 15 confirms this.

20. Parts (a) and (b) certainly. Part (c) can be worked by avoiding the given property, but it is likely that the property will also be used in (c), as underlined below.

$$\begin{aligned} 3\,(11 + 9) + 31 &= 3\,(20) + 31 \\ &= 3 \times (2 \times 10) + 31 \\ &= \underline{3 \times 2 \times 10 + 31} \end{aligned}$$

22. (c), (d)

23. (a) is not necessarily appropriate. The problem $3 + 5$ can be worked without it, but $23 + 15$ requires it in the step:

$$2 \text{ tens} + 3 \text{ tens} = (2 + 3) \text{ tens} = 5 \text{ tens}.$$

Chapter 5

1. 34

3. (a)

$$\begin{array}{|c|c|c|c|}
 & 5 & 3 & 4 \\
 & 3 & 2 & 2 \\
 & & 2 & 7 \\
2 & 1 & 6 & 9 \\
\hline
3 & 0 & 5 & 2
\end{array}$$

(b)

$$\begin{array}{|c|c|c|c|}
5 & 3 & 4 & 6 \\
- & 7 & 2 & 8 \\
\hline
4 & 6 & 1 & 8
\end{array}$$

(c)

$$\begin{array}{|c|c|c|c|}
 & 3 & 2 & 2 \\
 & \times & 2 & 7 \\
\hline
2 & 2 & 5 & 4 \\
6 & 4 & 4 & \\
\hline
8 & 6 & 9 & 4
\end{array}$$

(d)

$$\begin{array}{r|c|c|c|c|}
 & & 1 & 9 & 8 \\
\hline
27\,) & 5 & 3 & 4 & 6 \\
 & 2 & 7 & & \\
\hline
 & 2 & 6 & 4 & \\
 & 2 & 4 & 3 & \\
\hline
 & & 2 & 1 & 6 \\
 & & 2 & 1 & 6 \\
\hline
\end{array}$$

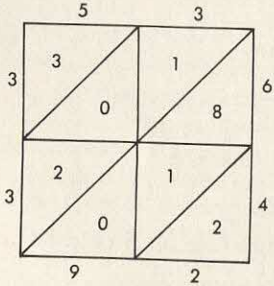
5. (a)
$$\begin{array}{r}
789 \\
673 \\
548 \\
\hline
\cancel{1}\cancel{8}\cancel{9}0 \\
2\cancel{0}1 \\
0
\end{array}$$

(b)
$$\begin{array}{r}
329 \\
762 \\
147 \\
\hline
1\cancel{1}\cancel{2}8 \\
23
\end{array}$$

(c)
$$\begin{array}{r}
652 \\
128 \\
543 \\
\hline
1\cancel{2}\cancel{1}3 \\
32
\end{array}$$

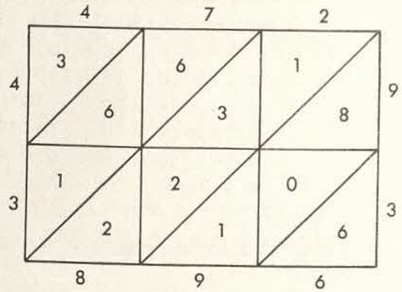
6. (a) $\left.\begin{array}{r} 82 \\ \underline{-46} \end{array}\right\}$ $\left.\begin{array}{r} 82+4 \\ -(46+4) \end{array}\right\}$ $\begin{array}{r} 86 \\ \underline{-50} \\ 36 \end{array}$ (b) $\left.\begin{array}{r} 126 \\ \underline{-98} \end{array}\right\}$ $\left.\begin{array}{r} 126+2 \\ -(98+2) \end{array}\right\}$ $\begin{array}{r} 128 \\ \underline{-100} \\ 28 \end{array}$

7. (a)



Answer: 3392

(b)



Answer: 43,896

9.
$$\begin{array}{r}
3 \\
\cancel{2}08 \\
\cancel{1}\cancel{8}\cancel{3} \\
2\cancel{0}\cancel{4}\cancel{2}6 \\
\cancel{1}\cancel{8}\cancel{7}\cancel{9}\cancel{7} \\
\cancel{2}\cancel{3}\cancel{8} \\
\cancel{2}\cancel{3}\cancel{8}
\end{array}$$

Answer: 23,086

11. How many 7's in 84? Arrange 84 things in rows of 7.

$$\begin{array}{l}\left.\begin{array}{c}o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\end{array}\right\}\text{ 10 sevens} \\ \qquad\qquad\qquad\qquad\quad \text{and} \qquad\qquad \text{make 12 sevens} \\ \left.\begin{array}{c}o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\end{array}\right\}\text{ 2 sevens}\end{array}$$

12. (a)

$$\left.\begin{array}{c}o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\ o\end{array}\right\}\text{ 7 twelves}$$

(b) Turn it on its side.

(c) If a set of 84 is divided into 7 subsets, how many will be in each subset?

(d) How many subsets of 12 are in a set of 84?

13.
```
o o o o o o o
o o o o o o o
o o o o o o o
o o o o o o o
o o o o o o o
o o o o o o o
o o o o o o o
o o o
```
This array shows that 7 subsets of 7 are in a set of 52, and that a subset of 3 members remains.

14. Let a and b represent positive numbers:

$$\begin{aligned} a + b &= (a + b) \\ a + (-b) &= a - b \text{ if } a \geqq b \\ &= -(b - a) \text{ if } b > a \\ (-a) + (-b) &= -(a + b) \\ (-a) + b &= -(a - b) \text{ if } a \geqq b \\ &= b - a \text{ if } b > a \end{aligned}$$

15. An example will illustrate: $38 \div 9$.

Subtract $\begin{array}{r} 38 \\ \underline{-9} \\ 29 \end{array}$ · Record a 1, to show 1 subtraction possible.

Subtract $\begin{array}{r} 29 \\ \underline{-9} \\ 20 \end{array}$ · Record a second 1.

Subtract	$\begin{array}{r} 20 \\ -9 \\ \hline 11 \end{array}$	Record a third 1.
Subtract	$\begin{array}{r} 11 \\ -9 \\ \hline \end{array}$	Record a fourth 1.
Remainder	2	Add 1's. Quotient is 4.

21. If we desire only natural numbers as answers, then (d) is required, since only if a > b will c be a natural number.

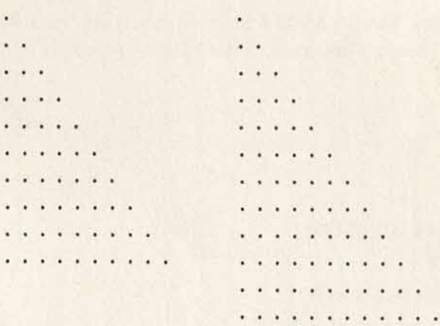
22. (b). Part (a) can be used if subtraction is introduced as a consequence of addition.

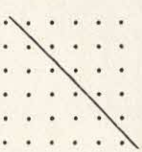
Chapter 6

1. Yes, the product of two even numbers is still a member of the set of even numbers.

3. Yes. The distributive property holds for all numbers which we encounter in arithmetic, and hence it would hold for the even numbers.

4. Yes. The product of any two odd numbers is also an odd number; the commutative and associative properties of multiplication hold for all numbers which are encountered in arithmetic, and hence they hold for the odd numbers.

6. 

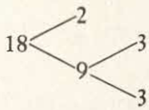
7. 

9. Yes. For example, 6 is both triangular and rectangular.

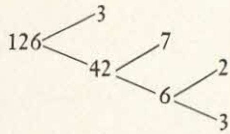
10. Yes. 36.

13. 23 and 29.

15. (a)



(f)



16. (a)

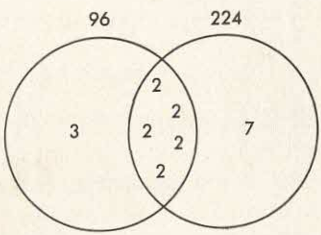
2)72
2)36
2)18
3)9
3

(g)

3)561
11)187
17

17. (c) $154 = 2 \cdot 7 \cdot 11$
$245 = 5 \cdot 7 \cdot 7$
greatest common factor: 7

18. (c)

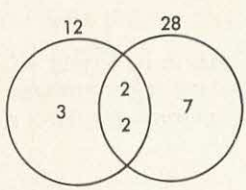


greatest common factor: $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

19. (b) The set of all factors of 45 is: $\{1, 3, 5, 9, 15, 45\}$
The set of all factors of 30 is: $\{1, 2, 3, 5, 6, 10, 15, 30\}$
greatest common factor: 15

20. (b) $12 = 2 \cdot 2 \cdot 3$
$30 = 2 \cdot 3 \cdot 5$
least common multiple: $2 \cdot 2 \cdot 3 \cdot 5 = 60$

21. (b)



least common multiple: $3 \cdot 2 \cdot 2 \cdot 7 = 84$

22. (c) Multiples of 6: $\{6, 12, 18, 24, 30, 36, \ldots\}$
Multiples of 10: $\{10, 20, 30, 40, \ldots\}$
least common multiple: 30

Chapter 7

1. (a) .403
   (b) 400.003
   (c) .003
   (d) .00403

2. (a) 14/100
   (b) 2831/1000
   (c) 7156002/100000
   (d) 400001/10000

3. (a) .4167 (approx.)
   (b) .2
   (c) .25
   (d) 2.375
   (e) 15.006

6. (a) $\dfrac{14+12+21}{28}=\dfrac{47}{28}$
   (b) $\dfrac{18+21+8}{48}=\dfrac{47}{48}$

9. (a) $-87/210$
   (b) 74/315

10. (a) $40/630=4/63$
    (b) $110/2520=11/252$
    (c) $112/112=1$
    (d) 288/65

11. (a) $\dfrac{5}{8}\times\dfrac{7}{9}=\dfrac{35}{72}$

13. (a) 5.688
    (b) 7.81

14. (a) 3.45
    (b) 644.4

15. (a) 413.424
    (b) There is a total of three decimal places in multiplier and multiplicand together. This many places appears in the product. The multiplier expresses hundredths; the multiplicand expresses tenths. The product must express thousandths.

16. (a) 1.21
    (b) 12.1
    (c) 23000
    (d) .0023

18. No. To test for equality, we must examine 29×43 and 41×31. They are not equal, hence the fractions are not equal.

20. (a) $[.101]_{\text{two}}$ means $\frac{1}{2}+\frac{0}{4}+\frac{1}{8}=\frac{5}{8}$

(b) $[.34]_{\text{five}}$ means $3 \times 5^{-1} + 4 \times 5^{-2}$ or $\frac{3}{5}+\frac{4}{25}$.

(c) $.625_{\text{ten}} = \frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\ldots$ (a start toward a binary representation

Multiply by 2.

$$1.25 = a + \frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\ldots$$

. . . For these to be equal, the whole number 1 must equal the whole number a, and the fraction .25 must equal the fraction

$$\frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\ldots$$

Therefore, $a = 1$, $.25 = \frac{b}{2}+\frac{c}{4}+\frac{d}{8}+\ldots$

Multiply by 2 again

$$.50 = b + \frac{c}{2}+\frac{d}{4}+\ldots$$

By the same reasoning, $b = 0$, and

$$.50 = \frac{c}{2}+\frac{d}{4}+\ldots$$

Multiply by 2.

$$1.0 = c + \frac{d}{2}+\ldots$$

Therefore: $c = 1$, and d and all successive fraction numerators must be zero.

We see then that

$$[.625]_{\text{ten}} = \frac{1}{2}+\frac{0}{4}+\frac{1}{8} = [.101]_{\text{two}}\,.$$

A comparable procedure would work to convert decimal fractions to quinary fractions.

23. (b) The fraction is decreased, so the decimal fraction is decreased.

25. (b)

26. (a)

28. (c)

Chapter 8

2. $-11, -9, -7, -5, -3$ (this is only one possibility)

3. (a) They are negatives (or opposites) of each other.
   (b) At least one of the numbers is zero.

4. (a) An integer which when divided by 2 gives an integer quotient is even.
   (b) An integer which is not even is odd.

6. (a) integers, rationals, reals
   (b) fractions (or rational numbers), reals
   (c) integers, rationals, reals
   (d) fractions (or rational numbers), reals
   (e) integers, rationals, reals

7. integers (and also to the rationals and reals)

9. The product is 1.999396. It differs by .000604. The products are 1.99996164, 2.00024449. The irrational number is $\sqrt{2}$.

12. (d)

14. (b), since $a \div b = 1$ in this case.

15. (c) Actually, zero is needed too.

16. (d)

Chapter 9

1. (d) Each names the same physical quantity—a particular length.

4. If $x = a^2$, $y = 3ab$ then
   $x^2 = (a^2)^2 = a^4$ Property V and Property III
   $y^2 = (3ab)^2 = 9a^2b^2$ Property V and Property III
   $x^2 + y = a^4 + 3ab$ Property IV
   $(x^2 + y) + y^2 = (a^4 + 3ab) + 9a^2b^2$ Property IV
   $x^2 + y + y^2 = a^4 + 3ab + 9a^2b^2$ Asso. Property of Addition

7. (a) It is not true that if $a \leq b$, then $b \leq a$.
   (b) It is not true that $a < a$.
   (c) It is not true that if $a \geqq b$, then $b \geqq a$.
   (d) It is not true that $a > a$.

8. Can touch the fingertips of (within a set of people).

10. Can run as fast as (within a set of boys).

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